SMA 209 : 2016 Final Exam
Guiding Solutions

1 (a) (i) $y' = x e^{-x^2/2}$

Integrate both sides w.r.t. x.

$$\int y' dx = \int x e^{-x^2/2} dx$$

$$\Rightarrow \int \frac{dy}{\cancel{dx}} \cancel{dx} = \int x e^{-x^2/2} dx$$

$$\Rightarrow \int dy = \int x e^{-x^2/2} dx$$

$$\Rightarrow y = \int x e^{-x^2/2} dx$$

To evaluate this integral, let $t = \frac{x^2}{2}$

$$\Rightarrow dt = \frac{\cancel{2} x \, dx}{\cancel{2}} = x \, dx$$

with this substitution we get:

$$y = \int e^{-t} dt = -e^{-t} + c$$

$$= -e^{-x^2/2} + c$$

$$\boxed{y = c - e^{-x^2/2}}$$

(ii)

$$y' = \sqrt{1-y^2}$$

Divide by $\sqrt{1-y^2}$:

$$\frac{y'}{\sqrt{1-y^2}} = 1$$

Integrate both sides w.r.t x :

$$\int \frac{\frac{dy}{\cancel{dx}}\,\cancel{dx}}{\sqrt{1-y^2}} = \int dx$$

$$\Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = x + c$$

For integrating L.H.S, let

~~tan~~ $y = \sin t \qquad t = \sin^{-1} y$

$$dy = \cos t \, dt$$

$$\Rightarrow \int \frac{\cos t \, dt}{\sqrt{1-\sin^2 t}} = x + c$$

$$\Rightarrow \int \frac{\cos t \, dt}{\sqrt{\cos^2 t}} = \int \frac{\cancel{\cos t}}{\cancel{\cos t}} dt = x + c$$

$$\therefore t = x + c$$

$$\sin^{-1} y = x + c \Rightarrow \boxed{y = \sin(x+c)}$$

1(b) $y' = -2xy \qquad y(0) = 2$

Divide by y:

$$\frac{y'}{y} = -2x$$

Integrate both sides w.r.t x

$$\int \frac{dy}{y} = -2 \int x\,dx$$

$$\ln|y| = -\cancel{2}\frac{x^2}{\cancel{2}} + C_1$$

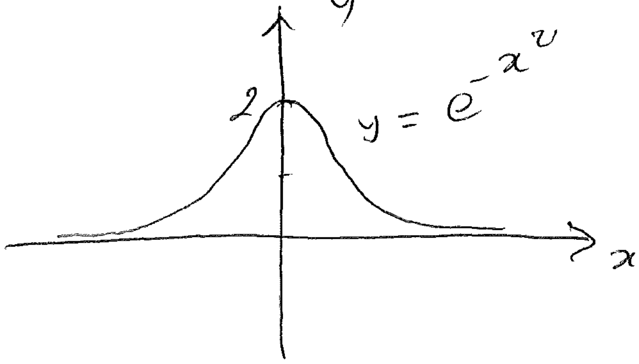
$$y = e^{-x^2+C_1} = c\,e^{-x^2}$$

where $c = e^{C_1}$

Given $y(0) = 2$

$$2 = c\,e^0 \Rightarrow c = 2$$

$$y = 2e^{-x^2}$$



1(c) Rate of change of volume w.r.t. p is

$$\frac{dV}{dp}$$

According to question;

$$\frac{dV}{dp} = -V/p$$

This is the required ODE.

To solve it, divide by V and then integrate both sides w.r.t p.

$$\frac{1}{V}\frac{dV}{dp} = -\frac{1}{p}$$

$$\Rightarrow \int \frac{1}{V}\frac{dV}{\cancel{dp}}\cancel{dp} = -\int \frac{dp}{p}$$

$$\int \frac{dV}{V} = -\int \frac{dp}{p}$$

$$\ln|V| = -\ln|p| + \cancel{\text{constant}}\ C_1$$

$$= \ln|p^{-1}| + \ln|C|$$

where $C_1 = \ln|C|$

$$\therefore \ln|V| = \ln\left|\frac{C}{p}\right| \Rightarrow V = \frac{C}{p}$$

or $\boxed{Vp = C}$

2. (a) $\sin x \cos y\, dx + \cos x \sin y\, dy = 0$

$M = \sin x \cos y \qquad N = \cos x \sin y$

$\frac{\partial M}{\partial y} = -\sin x \sin y \qquad \frac{\partial N}{\partial x} = -\sin x \sin y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and it is exact.

For Solution u

$u = \int M dx + k(y)$

$\Rightarrow \int \sin x\, dx \cos y + k(y)$

$\Rightarrow \cos y \int \sin x\, dx + k(y)$

$\Rightarrow \cos y\, x - \cos x + k(y)$

$= -\cos x \cos y + k(y)$

Now calculate

$\frac{\partial u}{\partial y} = -\cos x \cdot \sin y + k'(y)$

Equate it to N gives

$= \cancel{\cos x \sin y} + k'(y) = \cancel{\cos x \sin y}$

$k'(y) = 0 \Rightarrow k(y) = C_1$ (constant)

Thus we get : $u = \underline{-\cos x \cos y + C_1} = k$

$\Rightarrow \boxed{\cos x \cos y = C}$, $C = C_1 - k$

(b) $xy' = 2y + x^3 e^x$

This can be written as

$$xy' - 2y = x^3 e^x$$

Divide by x

$$\Rightarrow y' - \frac{2}{x} y = x^2 e^x$$

$$p(x) = -\frac{2}{x} \; ; \quad r(x) = x^2 e^x$$

$$h = \int p(x)\,dx = -2 \int \frac{dx}{x} = -2 \ln|x|$$
$$= \ln|x^{-2}|$$

$$e^h = e^{\ln(\frac{1}{x^2})} = \frac{1}{x^2} \qquad e^{-h} = e^{-\ln|\frac{1}{x^2}|}$$
$$= e^{\ln|x^2|}$$
$$= x^2$$

General Solution.

$$y = e^{-h}\left[\int e^h r(x)\,dx + c\right]$$

$$= x^2 \left[\int \frac{1}{\cancel{x^2}} \cancel{x^2} e^x\,dx + c\right]$$

$$= x^2 \left[\int e^x\,dx + c\right]$$

$$\boxed{y = x^2\left[e^x + c\right]}$$

(c) $y' = 3.2y - 10y^2$

Let $u = y^{1-a}$ here $a = 2$

$u = y^{1-2} = y^{-1}$

$u' = -1\, y^{-2}\, y'$

Substitute for y', y'

$= -y^{-2}\left[3.2y - 10y^2\right]$

$= -\left[\frac{3.2}{y} - 10\right]$

$= -\left[3.2u - 10\right]$

$\therefore$ Now ODE is linear in u

$u' + 3.2u = 10$ ~~the~~

Here $p(x) = 3.2$ $r(x) = 10$

$h = \int 3.2\,dx = 3.2x$

General Solution becomes

$u = e^{-h}\left[\int e^{h} r(x)\,dx + c\right]$

$= e^{-3.2x}\left[10\int e^{3.2x}\,dx + c\right]$

$= e^{-3.2x}\left[10\,\frac{e^{3.2x}}{3.2} + c\right]$

$$u = \left[\frac{10}{3.2} + c\, e^{-3.2x} \right]$$

$$\Rightarrow \frac{1}{y} = \left[\frac{10}{3.2} + c\, e^{-3.2x} \right]$$

$$\therefore y = \left[\frac{10}{3.2} + c\, e^{-3.2x} \right]^{-1}$$

3 (a) $$y' = \frac{y}{x} + \frac{2x^3 \cos x^2}{y}; \; y\left(\sqrt{\pi}\right) = 0$$

It is not in separable form, so we change the variable to

$$u(x) = \frac{y}{x}$$

$$\Rightarrow y = u(x)x$$

$$y' = u'(x)x + u(x)$$

Then the D. E. reduces to

$$u'(x)x + u(x) = u(x) + \frac{2x^2 \cos x^2}{u}$$

$$u'(x)x = \frac{2x^2 \cos x^2}{u}$$

$$uu'(x) = 2x \cos x^2$$

Integrating both sides with respect to x

$$\int u \, du = 2\int x \cos x^2 \, dx + c$$

use integration by substitution

$x^2 = t, \; 2x\,dx = dt$

$$\frac{u^2}{2} = \int \cos t \, dt + c$$

$$\frac{u^2}{2} = \sin t + c$$

$$\frac{u^2}{2} = \sin x^2 + c$$

$$u^2 = 2\sin x^2 + 2c$$

$$\frac{y^2}{x^2} = 2\sin x^2 + 2c$$

$$y^2 = 2x^2 \sin x^2 + 2cx^2$$

$$y = \pm\sqrt{2x^2 \sin x^2 + 2cx^2}$$

Given that $y\left(\sqrt{\pi}\right) = 0$

$$0 = \pm\sqrt{2\pi \sin \pi + 2c\pi}$$

$\sin \pi = 0$

$$0 = \pm\sqrt{2c\pi}$$

$$c = 0$$

$$y = \pm\sqrt{2x^2 \sin x^2} \quad \text{or} \quad y = \pm x\sqrt{2\sin x^2}$$

3 (b) $2x\tan y\,dx + \sec^2 y\,dy = 0$

$P\,(x, y) = 2x\tan y$

$Q\,(x, y) = \sec^2 y$

$\dfrac{\partial P}{\partial y} = 2x\sec^2 y\,, \qquad \dfrac{\partial Q}{\partial x} = 0$ Not exact.

Find R, $\quad R = \dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right)$

$\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x} = 2x\sec^2 y$

$\therefore\ \dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right) = \dfrac{1}{\sec^2 y}\left(2x\sec^2 y\right) = 2x$

Integrating factor, $\quad F = e^{\int R\,dx}$

$F = \exp\int 2x dx$

$F = e^{x^2}$

Multiply ODE by F to get

$2xe^{x^2}\tan y\,dx + e^{x^2}\sec^2 y\,dy = 0$

$M\,(x, y) = 2xe^{x^2}\tan y$

$N\,(x, y) = e^{x^2}\sec^2 y$

$\dfrac{\partial M}{\partial y} = 2xe^{x^2}\sec^2 y\ , \quad \dfrac{\partial N}{\partial x} = 2xe^{x^2}\sec^2 y$ It is now exact.

$u = \int M\,dx + k(y) = \int\left(2xe^{x^2}\tan y\right)dx + k(y)$

$u = \int\left(2xe^{x^2}\tan y\right)\,dx + k(y)$

use substitution, set $t = x^2$, $\dfrac{dx}{dt} = 2x$

$\therefore\ u = \int\left(\,e^t\tan y\,\right)\,dt + k(y)$

$\therefore\ u = e^t\tan y + k(y)$

$\therefore\ u = e^{x^2}\tan y + k(y)$

$\dfrac{\partial u}{\partial y} = e^{x^2}\sec^2 y + k'(y)$ This must be equal to N.

$$e^{x^2}\sec^2 y + k'(y) = e^{x^2}\sec^2 y$$

$$\therefore\ k'(y) = 0$$

$$k(y) = c' \qquad a\ constant$$

$$\Rightarrow e^{x^2}\tan y + c' = c'' \Rightarrow e^{x^2}\tan y = c\ ,\ c = c'' - c'$$

3 (c)

$$x^2 y'' - 5xy' + 9y = 0 \qquad\qquad y_1 = x^3$$

$$\Rightarrow y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0$$

$$p(x) = -\frac{5}{x}, \quad q(x) = \frac{9}{x^2}$$

$$U = \frac{1}{y_1^{\,2}}e^{-\int p(x)\,dx} = \frac{1}{(x^3)^2}e^{-\int p(x)\,dx} = \frac{1}{x^6}e^{5\int\frac{1}{x}dx} = \frac{1}{x^6}e^{5\ln|x|}$$

$$= \frac{1}{x^6}e^{\ln|x^5|} = \frac{x^5}{x^6} = \frac{1}{x}$$

Now evaluate $\int U\,dx = \int \frac{1}{x}\,dx = \ln|x|$

$$\therefore\ y_2 = y_1\int U\,dx = x^3\ln|x|$$

4 (a) $$T = \frac{z}{x^2+y^2}$$

$$\text{grad}\, T = \hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}$$

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left[z\,(x^2+y^2)^{-1}\right]$$

$$= z \times -1\,(x^2+y^2)^{-2} \times 2x$$

$$= -\frac{2xz}{(x^2+y^2)^2}$$

$$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left[z\,(x^2+y^2)^{-1}\right]$$

$$= z \times -1\,(x^2+y^2)^{-2} \times 2y$$

$$= -\frac{2yz}{(x^2+y^2)^2}$$

$$\frac{\partial T}{\partial z} = \frac{\partial}{\partial z}\left[z\,(x^2+y^2)^{-1}\right] = \frac{1}{(x^2+y^2)}$$

$$\therefore \text{grad}\, T = \frac{2xz}{(x^2+y^2)^2}\hat{i} + \frac{2yz}{(x^2+y^2)^2}\hat{j} + \frac{1}{(x^2+y^2)}\hat{k}$$

$$\text{grad } T\Big|_{P(0,1,2)} = \frac{2.0.2}{(1)^2} i + \frac{2.1.2}{1^2}\hat{j} + \frac{1}{1^2}\hat{k}$$

$$= 4\hat{j} + \hat{k}$$

This is the direction of maximum increase.

So the insect should fly in the opposite direction vector

$$= -4\hat{j} - k \quad \text{or} \quad \text{Unit vector} = -\frac{4j}{\sqrt{17}} - \frac{\hat{k}}{\sqrt{17}}$$

4 (b) $x^2 y'' - 4x\, y' + 6\, y = 0$

Comparing with

$x^2 y'' + axy' + b\, y = 0$

$a = -4,\ b = 6$

Characteristic equation

$m^2 + (a-1)m + b = 0$

$m^2 + -5m + 6 = 0$

$(m-3)(m-2) = 0$

Hence, two real roots, $m_1 = 3,\ m_2 = 2$

$y_1 = x^3, \qquad y_2 = x^2$

$\therefore\ y(x) = c_1 x^3 + c_2 x^2$

$y_1' = 3x^2$

$y_2' = 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$= x^3\, 2x\ -\ x^2\, 3x^2$

$= 2x^4 - 3x^4$

$\therefore\ W = -x^4$

Wronskian is non-zero, except at $x = 0$. Hence these y_1 and y_2 form basis of solutions.

Does this mean that these solutions do not form the basis of solutions?

No.

4 (c) $y'' + 6y' + 9y = 18\cos 3x$

Homogeneous solution is obtained from the characteristic equation:

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda + 3)(\lambda + 3) = 0$

Double roots, $\lambda_1 = \lambda_2 = -3$

$y_1 = e^{-3x}$, $y_2 = xe^{-3x}$

$y_h = c_1 e^{-3x} + c_2 x e^{-3x}$

$y_h = e^{-3x}(c_1 + c_2 x)$

For y_p, we set

$y_p = a\cos 3x + b\sin 3x$

$y_p' = -3a\sin 3x + 3b\cos 3x$

$y_p'' = -9a\cos 3x - 9b\sin 3x$

$= -9\,y_p$

Substituting we get

$-9y + 6(-3a\sin 3x + 3b\cos 3x) + 9y = 18\cos 3x$

$-18a\sin 3x + 18b\cos 3x = 18\cos 3x$

$\therefore\ a = 0,\ b = 1$

$\therefore\ y_p = \sin 3x$

$y = y_h + y_p = e^{-3x}(c_1 + c_2 x) + \sin 3x$

5 (a) $y_1' = y_2$ $\qquad y_1(0) = 1, \quad y_2(0) = 0$

$y_2' = y_1$

In the matrix form it can be written as:

$$y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ay}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\Rightarrow \lambda_1 = 1, \lambda_2 = -1$$

Now determine the eigenvectors:

For $\lambda_1 = 1$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$-x_1 + x_2 = 0$ $\qquad \Rightarrow x_1 = x_2$ $\qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_1 - x_2 = 0$

For $\lambda_2 = -1$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 + x_2 = 0$ $\qquad \Rightarrow x_1 = -x_2$ $\qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

General solution:

$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + \ldots + c_n \mathbf{x}^{(n)} e^{\lambda_n t}$$

$$y = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

$$y_1 = c_1 e^t + c_2 e^{-t}$$

$$y_2 = c_1 e^t - c_2 e^{-t}$$

Initial value problem:

given $y_1(0) = 1, \qquad y_2(0) = 0$

$$y_1 = c_1 + c_2 = 1$$

$$y_2 = c_1 - c_2 = 0$$

$$\therefore \quad c_1 = c_2 = \frac{1}{2}$$

$$\therefore \quad y_1 = \frac{1}{2}\left(e^t + e^{-t}\right) = \cosh t$$

$$y_2 = \frac{1}{2}\left(e^t - e^{-t}\right) = \sinh t$$

5 (b)

$$f(x) = f(x+p)$$

$$f(ax) = f(ax+p) = f\left(a\left(x+\frac{p}{a}\right)\right) \Rightarrow \text{period } \frac{p}{a}$$

5(c) $f(x) = x$

$f(-x) = -x = -f(x)$ $\quad f(x+2\pi) = f(x)$

This is an odd funtion and hence the Fourier series will only be a sine series

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_1 = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

Integrate by parts

$$= \frac{2}{\pi}\left\{ \left[x \times -\frac{\cos nx}{n} \right]_0^{\pi} - \int_0^{\pi} 1 \cdot \times -\frac{\cos nx}{n} dx \right\}$$

$$= \frac{2}{\pi}\left[-\frac{\pi \cos n\pi}{n} + \frac{1}{n}\int_0^{\pi} \cos nx \, dx \right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n} \cos n\pi + \frac{1}{n} \underbrace{\left[\frac{\sin nx}{n} \right]_0^{\pi}}_{\text{zero}} \right]$$

$$= -\frac{2 \cancel{\pi}}{\cancel{\pi}} \frac{\cos n\pi}{n} = -\frac{2 \cos n\pi}{n}$$

$$b_1 = -\frac{2 \cos \pi}{1} = 2 \qquad \cos \pi = -1$$

$$b_2 = -\frac{2 \cos 2\pi}{2} = -\frac{2}{2}$$

$$b_3 = -\frac{2\cos 3\pi}{3} = -\frac{2\cos\pi}{3} = \frac{2}{3}$$

~~b_4~~

Thus the Fourier Series:

$$x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$$

$$= -2\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} \sin nx$$

6(a) $f(x,y,z) = 2x^2 + 3y^2 + z^2$

$\text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$\frac{\partial f}{\partial x} = 4x \qquad \frac{\partial f}{\partial y} = 3\cdot 2\cdot y = 6y$

$\frac{\partial f}{\partial z} = 2z$

$\text{grad } f = [4x, 6y, 2z]$

$\text{grad } f \Big|_{P:(2,1,3)} = [4\times 2, 6\times 1, 2\times 3]$

$= [8, 6, 6]$

Vector $\underline{a} = [1, 0, -2]$

Make it a unite vector

$\hat{a} = \frac{\underline{a}}{\|\underline{a}\|}$; $\|\underline{a}\| = \sqrt{1+4} = \sqrt{5}$

$\hat{a} = \frac{1}{\sqrt{5}}[1, 0, -2]$

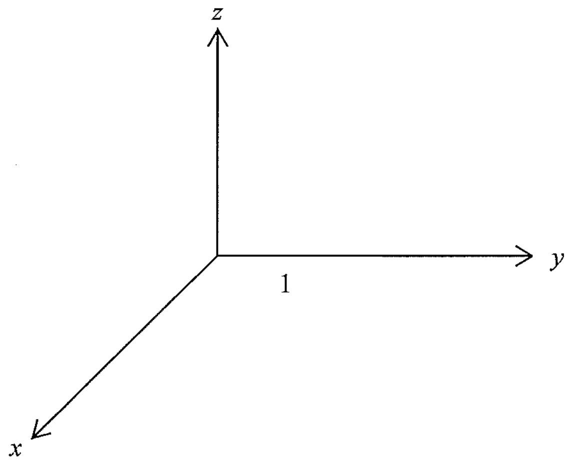
$\therefore D_a f(P) = \hat{a}\cdot \text{grad } f\Big|_P = \frac{1}{\sqrt{5}}[1,0,-2]\cdot[8,6,6]$

$= \frac{1}{\sqrt{5}}[8-12] = -\frac{4}{\sqrt{5}}$

That f decreases in the direction of $\underline{a}$

6 (b) $\mathbf{r}(t) = [t, \cos t, \sin t]$; $0 \le t \le 2\pi$

(i) $y^2 + z^2 = 1$



(ii) $\mathbf{r}'(t) = [1, -\sin t, \cos t]$

$\mathbf{r}'(t)\,\mathbf{r}'(t) = 1 + \sin^2 t + \cos^2 t$

$= 2$

$|\mathbf{r}'(t)| = \sqrt{\mathbf{r}'(t) \bullet \mathbf{r}'(t)} = \sqrt{2}$

6 (c)

$$\underline{F} = x^2y^2z^2[x, y, z]$$

$$\operatorname{div} \underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$F_1 = x^3y^2z$ $\quad$ $F_2 = x^2y^3z^2$ $\quad$ $F_3 = x^2y^2z^3$

$\dfrac{\partial F_1}{\partial x} = 3x^2y^2z^2$ $\quad$ $\dfrac{\partial F_2}{\partial y} = 3x^2y^2z^2$; $\quad$ $\dfrac{\partial F_3}{\partial z} = 3x^2y^2z^2$

$$\operatorname{div} \underline{F} = 3x^2y^2z^2 + 3x^2y^2z^2 + 3x^2y^2z^2$$
$$= 9x^2y^2z^2$$

$$\operatorname{div} \underline{F}\Big|_{P:(3,-1,4)} = 9(3)^2(-1)^2(4)^2$$
$$= 9 \cdot 9 \cdot 1 \cdot 16 = 81 \times 16 = 1296$$

6(d) $\underline{v} = [x, y, -z]$

$$\text{Curl } \underline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & -z \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -z \end{vmatrix} - \hat{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & -z \end{vmatrix} + \hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{vmatrix}$$

$$= \hat{i}\left(-\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - j\left|-\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right| + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= \underline{0}$$

Curl $\underline{v} = \underline{0}$

$\therefore$ $\underline{v}$ is an irrotational vector field

and the flow is irrotational.