SMA102 Guiding Solutions Final Exam 2017

Question 1 (a)

$\int x^2 e^{2x}\, dx$

Integration by parts:

$$\int u\, dv = uv - \int v\, du$$

We let

$$u = x^2, \qquad \therefore \frac{du}{dx} = 2x$$

$$dv = e^{2x}\, dx \qquad \therefore v = \frac{1}{2}e^{2x}$$

Then

$$\int x^2 e^{2x}\, dx = \frac{1}{2}x^2 e^{2x} - \int \frac{1}{2}e^{2x}\, 2x\, dx$$

$$= \frac{1}{2}x^2 e^{2x} - \int e^{2x} x\, dx \qquad \_\_\_\_\_\_\_(1)$$

Integrate by parts once again. We let

$$u = x, \qquad \therefore \frac{du}{dx} = 1$$

$$dv = e^{2x}\, dx \qquad \therefore v = \frac{1}{2}e^{2x}$$

Then

$$\int e^{2x} x\, dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\, dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\, dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c \qquad \_\_\_\_\_\_\_(2)$$

Substituting (2) into (1).

$$\int x^2 e^{2x}\, dx = \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]$$

$$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$$

ii) $\int \sqrt{x}\ln x| dx$

Integrate by parts, let $u = \ln x \qquad du = \frac{1}{x}$

$$dv = x^{1/2} \qquad v = \frac{2}{3}x^{3/2}$$

$$I = \int \sqrt{x}\ln x\, dx = \ln x\, \frac{2}{3}x^{3/2} - \int \frac{2}{3}x^{3/2} \cdot \frac{1}{x}\, dx$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2} dx$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\left[\frac{2}{3}x^{3/2}\right] + C$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$$

$$I = \frac{2}{3}x^{3/2}\left(\ln x - \frac{2}{3}\right) + C$$

(b) $\int \frac{2x+4}{x^3-2x^2}\,dx$,

$$\frac{2x+4}{x^2(x-2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2} \quad \text{----(1)}$$

Multiplying through (1) by $x^2(x-2)$

$2x+4=x(x-2)A+(x-2)B+x^2C$ ----- (2) solve for the constants *A*, *B*, and *C*.

Put x=0 in (2), $4=(0-2)B$, $B=-2$

Put x=2 in (2), $8=4C$, $C=2$.

No direct substitution for x can help find A, so we have to expand the RHS of (2) and compare coefficients. That is $2x+4=Ax^2-2Ax+Bx-2B+Cx^2$

$2x+4=(A+C)x^2+(B-2A)x-2B$ ----- (3). From (3)

$2x=(B-2A)x \rightarrow 2=(B-2A) \rightarrow 2=(-2-2A)$, $A=-2$

Substituting the values of *A*, *B*, and *C* in (1) yields the partial fraction decomposition as

$$\frac{2x+4}{x^2(x-2)}=-\frac{2}{x}-\frac{2}{x^2}+\frac{2}{x-2}$$

Thus $\int \frac{2x+4}{x^3-2x^2}\,dx=-2\int\frac{1}{x}dx-2\int\frac{1}{x^2}dx+2\int\frac{1}{x-2}dx$

$$\int \frac{2x+4}{x^3-2x^2}\,dx=-2\ln|x|+\frac{2}{\mathrm{x}}+2\ln|x-2|+c$$
$$=\frac{2}{\mathrm{x}}+2\ln\left|\frac{x-2}{x}\right|+c$$

(c) $\int_1^{ac}\frac{1}{t}dt=[\ln t]_1^{ac}=[\ln ac]-[\ln 1]=[\ln a+\ln c]-[\ln 1]$

$$=2+5-0=7$$

Question 2 (a)

(i)

$$\int \tan(4x)\sec^4(4x)dx = \int \tan(4x)\sec^2(4x)\sec^2(4x)dx = \int \tan 4x(1+\tan^2 4x)\sec^2 4xdx$$

$$= \int \tan 4x \sec^2 4xdx + \int \tan^3 4x \sec^2 4xdx$$

$$Let\ t = 4x,\quad dt = 4dx$$

$$= \frac{1}{4}\int \tan t \sec^2 tdt + \frac{1}{4}\int \tan^3 t \sec^2 tdx$$

$$t = \tan t,\quad du = \sec^2 tdt$$

$$= \frac{1}{4}\int udu + \frac{1}{4}\int u^3 du = \frac{1}{4}\frac{u^2}{2} + \frac{1}{4}\frac{u^4}{4} + c = \frac{1}{8}\tan^2 t + \frac{1}{16}\tan^4 t + c$$

$$= \frac{1}{8}\tan^2 4x + \frac{1}{16}\tan^4 4x + c$$

OR

$$\int \tan(4x)\sec^4(4x)dx = \int \tan(4x)\sec(4x)\sec^3(4x)dx$$

$$t = \sec 4x,$$
$$dt = 4\tan 4x \sec 4xdx$$

$$= \frac{1}{4}\int t^3 dt = \frac{1}{4}\left[\frac{t^4}{4}\right] + c = \frac{1}{16}\sec^4 4x + c$$

(ii)

$$\int (\ln x)^2 dx$$

Integrate it by parts by considering first function $u = (\ln x)^2$ and second function $dv = dx$. Then we get

$$\int (\ln x)^2 dx = (\ln(x))^2 x - 2\int \ln x \cdot \frac{1}{x} dx$$

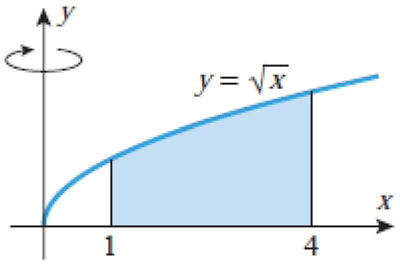
$$= x[\ln(x)]^2 - 2\int \ln x\, dx$$

continue integration by parts

$$= x(\ln x)^2 - 2\ln x \; x + 2\int \frac{1}{x} x\, dx$$

$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

(b)



$$V = \int_1^4 2\pi x\sqrt{x}\,dx = 2\pi\int_1^4 x^{3/2}\,dx = \left[2\pi\cdot\frac{2}{5}x^{5/2}\right]_1^4 = \frac{4\pi}{5}[32-1] = \frac{124\pi}{5}$$

(c)

Arc length $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$

$y = 3x^{3/2} - 1$ from $x = 0$ to 1

$\therefore\ y' = \frac{3}{2}\cdot 3x^{1/2}$

$= \frac{9}{2}x^{1/2}$

$L = \int_0^1 \sqrt{1+\left[\frac{9}{2}x^{1/2}\right]^2}\,dx$

$= \int_0^1 \sqrt{1+\frac{81}{4}x}\,dx$

Let $u = 1+\frac{81}{4}x$, $\frac{du}{dx} = \frac{81}{4}$, $du = \frac{81}{4}dx$, $dx = \frac{4}{81}du$

$x = 1,\ u = \frac{85}{4}$

$x = 0,\ u = 1$

$L = \frac{4}{81}\int_1^{85/4} u^{1/2}du = \frac{4}{81}\left[\frac{u^{3/2}}{3/2}\right]_1^{85/4} = \frac{8}{243}\left[u^{3/2}\right]_1^{85/4}$

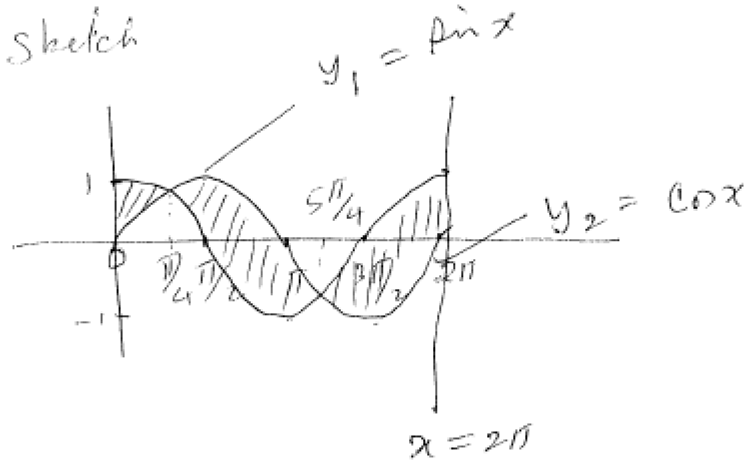
$= \frac{8}{243}\left[\frac{85}{4}\sqrt{\frac{85}{4}} - 1\right]$

$= \frac{1}{243}\left[85\sqrt{85} - 8\right]$

$L \approx 3.19$ units

Question 3

(a)



The area to be calculated is the shaded area, which can be founds as:

$$\int_0^{\pi/4}(\cos x - \sin x)\,dx + \int_{\pi/4}^{3\pi/4}(\sin x - \cos x)\,dx + \int_{3\pi/4}^{2\pi}(\cos x - \sin x)\,dx$$

$$= \left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4} + \left[\sin x + \cos x\right]_{5\pi/4}^{2\pi}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - [0+1] + \left\{\left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] - \left\{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right\}\right\}$$

$$+ \left[\{0 + 1\} - \left\{-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right\}\right]$$

$$= 2/\sqrt{2} - 1 - \left\{-\frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}}\right\} + \left[1 + \frac{2}{\sqrt{2}}\right]$$

$$= \frac{2}{\sqrt{2}} - 1 + \frac{4}{\sqrt{2}} + 1 + \frac{2}{\sqrt{2}} = \frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{2}$$

$$= 4\sqrt{2} \text{ units}$$

(b) $u_k = \dfrac{(-1)^k x^{2k+1}}{2^{2k+1}(k!)(k+1)!}$ $u_{k+1} = \dfrac{(-1)^{k+1} x^{2(k+1)+1}}{2^{2(k+1)+1}(k+1)!(k+1+1)!}$

$$\rho = \lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\frac{x^{2k+3}}{2^{2k+3}(k+1)!(k+2)!}\times\frac{2^{2k+1}(k!)(k+1)!}{x^{2k+1}}$$

$$\rho = \lim_{k\to\infty}\frac{x^{2k}x^3}{2^{2k+3}(k+1)k!(k+2)(k+1)!}\times\frac{2^{2k+1}(k!)(k+1)!}{x^{2k}x^1} = \lim_{k\to\infty}\frac{x^2}{2^2(k+1)(k+2)}$$

$= \lim_{k\to\infty}\dfrac{x^2}{4[(k^2+3k+2)]} = 0$, converges absolutely, hence converges.

(c) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{5^k} - \frac{1}{k(k+1)}\right)$

This can be written as the sum of two series:

$$= \sum_{k=1}^{\infty}\frac{1}{5^k} - \sum_{k=1}^{\infty}\frac{1}{k(k+1)}$$

Both are geometric series.

Series (1).

$\displaystyle\sum_{k=1}^{\infty}\frac{1}{5^k}$ is a geometric series with $a = \dfrac{1}{5}$, $r = \dfrac{1}{5}$

$$\therefore S = \frac{a}{1-r} = \frac{1/5}{1-1/5} = \frac{1}{4}$$

Series (2).

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$S_n = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1 + \left(-\frac{1}{2}+\frac{1}{2}\right) + \left(-\frac{1}{3}+\frac{1}{3}\right) + \left(-\frac{1}{4}+\frac{1}{4}\right) + \cdots + \left(-\frac{1}{n}+\frac{1}{n}\right) - \frac{1}{n+1}$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

$$\therefore \text{ Total sum} = \frac{1}{4} - 1 = -\frac{3}{4}$$

Question 4(a)

$$f(x) = \frac{1}{x} = x^{-1} \qquad f(-1) = \frac{1}{(-1)} = -1$$

$$f'(x) = -\frac{1}{x^2} \qquad f'(-1) = -\frac{1}{(-1)^2} = -1$$

$$f''(x) = \frac{2.1}{x^3} \qquad f''(x) = \frac{2.1}{(-1)^3} = -2!$$

$$f'''(x) = -\frac{3.2.1}{x^4} \qquad f'''(x) = -\frac{3.2.1}{(-1)^4} = -3!$$

$$f(x) = -1 - (x-(-1)) + \frac{-2!}{2!}(x-(-1))^2 - \frac{3!}{3!}(x-(-1))^3 + \ldots$$

$$f(x) = (-1)[1 + (x+1) + (x+1)^2 + (x+1)^3 + \ldots] = \sum_{k=0}^{\infty}(-1)(x+1)^k$$

(b) $\int_3^{+\infty} \frac{2}{x^2-1}dx \qquad \frac{2}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} \qquad 2 = A(x-1) + B(x+1)$

$A = -1,\ B = 1$

$$\lim_{b\to+\infty}\int_3^b \frac{1}{x-1}dx - \lim_{b\to+\infty}\int_3^b \frac{1}{x+1}dx$$

$$\lim_{b\to+\infty}\left[\ln(x-1)\right]_3^b - \lim_{b\to+\infty}\left[\ln(x+1)\right]_3^b$$

$$\lim_{b\to+\infty}\left[\ln(b-1) - \ln(2)\right] - \lim_{b\to+\infty}\left[\ln(b+1) - \ln(4)\right]$$

$$\infty - \infty - \ln(2) + \ln(4) = \ln(4/2) = \ln 2$$

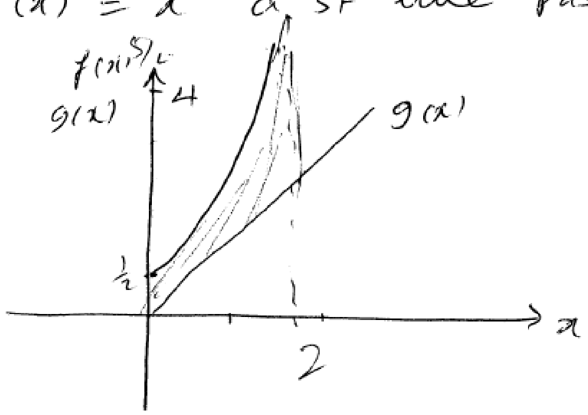
(c)

$f(x) = \frac{1}{2} + x^2$ $\quad g(x) = x$

$f(x=0) = \frac{1}{2}$

$f(x=1) = 3/2$ $\quad f(x=2) = 4+\frac{1}{2} = \frac{9}{2}$

$g(x) = x$ a st line passing thugh origin



The volume of the solid generated:

$$V = \pi \int_a^b \left[[f(x)]^2 - (g(x))^2 \right] dx \qquad a=0,\ b=2$$

$$= \pi \int_0^2 \left[\left(\frac{1}{2} + x^2\right)^2 - x^2 \right] dx$$

$$= \pi \int_0^2 \left[\frac{1}{4} + x^2 + x^4 - x^2 \right] dx = \pi \int_0^2 \left[\frac{1}{4} + x^4 \right] dx$$

$$= \pi \left[\frac{x}{4} + \frac{x^5}{5} \right]_0^2 = \pi \left[\frac{2}{4} + \frac{2^5}{5} \right] = \pi \left[\frac{1}{2} + \frac{32}{5} \right]$$

$$= \frac{(5+64)}{10} \pi = \frac{69\pi}{5} \text{ Units}$$

Question 5(a) i)

Standard matrix can be obtained by writing equations in matrix form as:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \mathbf{w} = A\mathbf{x}$$

Then the standard matrix A is:

$$A = \begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$$

(ii)

Orthogonal vectors if $\mathbf{u} \cdot \mathbf{v} = 0$

This gives

$$k^2 + 5k + 6 = 0$$

$$(k+3)(k+2) = 0$$

$\Rightarrow k = -3$, and $k = -2$

Therefore for $k = -3$ and $k = -2$, vectors will be orthogonal.

(b)

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} + c_1\mathbf{u_4} = (0, 5, 6, -3)$$

$$\Rightarrow c_1(-1, 3, 2, 0) + c_2(2, 0, 4 -1) + c_3(7, 1, 1, 4) + c_4(6, 3, 1, 2) = (0, 5, 6, -3)$$

This gives:

$$-c_1 + 2c_2 + 7c_3 + 6c_4 = 0$$

$$3c_1 + \quad c_3 + 3c_4 = 5$$

$$2c_1 + 4c_2 + c_3 + c_4 = 6$$

$$-c_2 + 4c_3 + 2c_4 = -3$$

The augmented matrix is obtained as:

$$\begin{bmatrix} -1 & 2 & 7 & 6 & 0 \\ 3 & 0 & 1 & 3 & 5 \\ 2 & 4 & 1 & 1 & 6 \\ 0 & -1 & 4 & 2 & -3 \end{bmatrix}$$

Add row 3 to to row 1:

$$\begin{bmatrix} 1 & 6 & 8 & 7 & 6 \\ 3 & 0 & 1 & 3 & 5 \\ 2 & 4 & 1 & 1 & 6 \\ 0 & -1 & 4 & 2 & -3 \end{bmatrix}$$

Add −3 times row 1 to row 2 and −2 times row 1 to row 3:

$$\begin{bmatrix} 1 & 6 & 8 & 7 & 6 \\ 0 & -18 & -23 & -18 & -13 \\ 0 & -8 & -15 & -13 & -6 \\ 0 & -1 & 4 & 2 & -3 \end{bmatrix}$$

Swap −1 times row 4 with row 2:

$$\begin{bmatrix} 1 & 6 & 8 & 7 & 6 \\ 0 & 1 & -4 & -2 & 3 \\ 0 & -8 & -15 & -13 & -6 \\ 0 & -18 & -23 & -18 & -13 \end{bmatrix}$$

Add 18 times row 2 to row 4 and 8 times row 2 to row 3:

$$\begin{bmatrix} 1 & 6 & 8 & 7 & 6 \\ 0 & 1 & -4 & -2 & 3 \\ 0 & 0 & -47 & -29 & 18 \\ 0 & 0 & -95 & -54 & 41 \end{bmatrix}$$

Multiply row 3 by 2 and multiply row 4 by −1:

$$\begin{bmatrix} 1 & 6 & 8 & 7 & 6 \\ 0 & 1 & -4 & -2 & 3 \\ 0 & 0 & -94 & -58 & 36 \\ 0 & 0 & 95 & 54 & -41 \end{bmatrix}$$

Add row 4 to row 3:

$$\begin{bmatrix} 1 & 6 & 8 & 7 & 6 \\ 0 & 1 & -4 & -2 & 3 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 95 & 54 & -41 \end{bmatrix}$$

Subtract 95 times row 3 from row 4:

$$\begin{bmatrix} 1 & 6 & 8 & 7 & 6 \\ 0 & 1 & -4 & -2 & 3 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 434 & 434 \end{bmatrix}$$

Divide row 4 by 434:

$$\begin{bmatrix} 1 & 6 & 8 & 7 & 6 \\ 0 & 1 & -4 & -2 & 3 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Equating coefficients gives:

$$c_1 + 6c_2 + 8c_3 + 7c_4 = 6 \qquad \text{\_\_\_\_\_\_\_\_(1)}$$
$$c_2 - 4c_3 - 2c_4 = 3 \qquad \text{\_\_\_\_\_\_\_\_(2)}$$
$$c_3 - 4c_4 = -5 \qquad \text{\_\_\_\_\_\_\_\_(3)}$$
$$c_4 = 1 \qquad \text{\_\_\_\_\_\_\_\_(4)}$$

Substituting $c_4 = 1$ in (3) we get:

$$c_3 - 4 = -5$$
$$c_3 = -1$$

Substituting in (2) we get:

$$c_2 + 4 - 2 = 3$$
$$c_2 = 1$$

Substituting in (1) we get:

$$c_1 + 6 - 8 + 7 = 6$$
$$c_1 = 1$$

$\therefore\ c_1 = 1,\ c_2 = 1,\ c_3 = -1$ and $c_4 = 1$

(c)

Determine the inner products

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = -\frac{12}{25} + \frac{12}{25} = 0$$

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0 + 0 + 0 = 0$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0 + 0 + 0 = 0$$

$$\|\mathbf{v}_1\| = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle^{1/2} = \left(\frac{9}{25} + \frac{16}{25}\right)^{1/2} = 1$$

$$\|\mathbf{v}_2\| = \langle \mathbf{v}_2, \mathbf{v}_2 \rangle^{1/2} = \left(\frac{16}{25} + \frac{9}{25}\right)^{1/2} = 1$$

$$\|\mathbf{v}_3\| = \langle \mathbf{v}_3, \mathbf{v}_3 \rangle^{1/2} = 1$$

Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form an orthonormal set

Question 6(a)

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

Reduce it to reduced row-echelon form:

Row 2 − 5 × Row 1

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 7 & -6 & 2 \end{bmatrix}$$

Row 3 − 7 × Row 1

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix}$$

Row 3 − Row 2

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

Row 1 + Row 2

$$A = \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

This is in reduced row-echelon form.

The solution for $A\mathbf{x} = \mathbf{0}$ is

$$x_1 - 16x_3 = 0$$
$$x_2 - 19x_3 = 0$$

Set $x_3 = t,\ x_1 = 16t,\ x_2 = 19t.$

The null space basis then is $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(b)

$\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (-1, 1, 0)$, $\mathbf{u}_3 = (1, 2, 1)$

$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = -1 + 1 + 0 = 0$

$\langle \mathbf{u}_1, \mathbf{u}_3 \rangle = 1 + 2 + 1 = 4 \neq 0$

$\langle \mathbf{u}_2, \mathbf{u}_3 \rangle = -1 + 2 + 0 = 1 \neq 0$

These vectors are not orthogonal.

Let us assume that the orthogonal vectors are $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

Step 1: Set $\mathbf{v}_1 = \mathbf{u}_1$

Step 2: $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \mathbf{u}_2$, since $\langle \mathbf{u}_2, \mathbf{v}_1 \rangle = 0$

Step 3:

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

$$= \mathbf{u}_3 - \frac{4}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{1}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

$$\|\mathbf{v}_1\|^2 = 1 + 1 + 1 = 3$$

$$\|\mathbf{v}_2\|^2 = 1 + 1 = 2$$

$$\therefore \mathbf{v}_3 = \mathbf{u}_3 - \frac{4}{3}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2$$

$$= (1, 2, 1) - \frac{4}{3}(1, 1, 1) - \frac{1}{2}(-1, 1, 0)$$

$$= \left(\left(1 - \frac{4}{3} + \frac{1}{2}\right), \left(2 - \frac{4}{3} - \frac{1}{2}\right), \left(1 - \frac{4}{3}\right)\right)$$

$$= \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$$

These form an orthogonal set.

$\therefore \mathbf{v}_1 = (1, 1, 1)$,

$\mathbf{v}_2 = (-1, 1, 0)$,

$\mathbf{v}_3 = \left(\dfrac{1}{6}, \dfrac{1}{6}, -\dfrac{1}{3}\right)$

(c) $A=\begin{bmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{bmatrix}$

The characteristic equation is

$$(\lambda I - A)=\begin{vmatrix} \lambda-6 & -2\sqrt{3} \\ -2\sqrt{3} & \lambda-7 \end{vmatrix}=0$$

$$(\lambda-6)(\lambda-7)-12=\lambda^2-13\lambda+30=0$$

$\lambda=3,\ 10$

When $\lambda=3$

$$(3I-A)=\begin{pmatrix} 3-6 & -2\sqrt{3} \\ -2\sqrt{3} & 3-7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} -3 & -2\sqrt{3} \\ -2\sqrt{3} & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-3x_1-2\sqrt{3}x_2=0$, $-2\sqrt{3}x_1-4x_2=0$, if $x_2=s$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} -2/\sqrt{3}s \\ s \end{pmatrix}=s\begin{pmatrix} -2/\sqrt{3} \\ 1 \end{pmatrix}$. $|\mathbf{x}_1|=\sqrt{(-2/\sqrt{3})^2+(1)^2}=\sqrt{\dfrac{7}{3}}$ Normalizing the vectors

$$\mathbf{u}_1=\frac{\mathbf{x}_1}{|\mathbf{x}_1|}=\begin{pmatrix} -2/\sqrt{7} \\ \sqrt{3}/\sqrt{7} \end{pmatrix},$$

When $\lambda=10$

$$(10I-A)=\begin{pmatrix} 10-6 & -2\sqrt{3} \\ -2\sqrt{3} & 10-7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 4 & -2\sqrt{3} \\ -2\sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$4x_1-2\sqrt{3}x_2=0$, $-2\sqrt{3}x_1+3x_2=0$, if $x_2=s$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2}s \\ s \end{pmatrix}=s\begin{pmatrix} \frac{\sqrt{3}}{2} \\ 1 \end{pmatrix}.\ |\mathbf{x}_1|=\sqrt{(\sqrt{3}/2)^2+(1)^2}=\sqrt{\frac{7}{4}}=\frac{1}{2}\sqrt{7}$$

$$\mathbf{u}_2=\frac{\mathbf{x}_2}{|\mathbf{x}_2|}=\begin{pmatrix} \sqrt{3}/\sqrt{7} \\ 2/\sqrt{7} \end{pmatrix}$$

$$P = \begin{bmatrix} -2/\sqrt{7} & \sqrt{3}/\sqrt{7} \\ \sqrt{3}/\sqrt{7} & 2/\sqrt{7} \end{bmatrix},$$

$$P^{-1} = \frac{1}{-1}\begin{bmatrix} 2/\sqrt{7} & -\sqrt{3}/\sqrt{7} \\ -\sqrt{3}/\sqrt{7} & -2/\sqrt{7} \end{bmatrix} = \begin{bmatrix} -2/\sqrt{7} & \sqrt{3}/\sqrt{7} \\ \sqrt{3}/\sqrt{7} & 2/\sqrt{7} \end{bmatrix},$$

$$AP = \begin{bmatrix} {}^{-6}\!/_{\sqrt{7}} & {}^{10\sqrt{3}}\!/_{\sqrt{7}} \\ {}^{3\sqrt{3}}\!/_{\sqrt{7}} & {}^{20}\!/_{\sqrt{7}} \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} {}^{21}\!/_{7} & 0 \\ 0 & {}^{70}\!/_{7} \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}$$