Charles Darwin University

Final Examination

Family Name	
Given Name/s	
Student Number	
Teaching Period	Semester 1, 2018

SMA209 – Mathematics 2A	DURATION	
	Reading Time:	10 minutes
	Writing Time:	180 minutes

INSTRUCTIONS TO CANDIDATES

1 Answer all **six** questions.
2 All questions are of equal value, and parts carry marks as indicated.
3 Read **ALL** questions carefully.
4 Show all working neatly in all parts. Answers without working details will attract little marks.
5 All symbols, unless stated otherwise, have their usual meanings.

EXAM CONDITIONS

You may begin writing from the commencement of the examination session. The reading time indicated above is provided as a guide only.

This is a CLOSED BOOK examination

Any non-programmable calculator is permitted

No handwritten notes are permitted

No dictionaries are permitted

ADDITIONAL AUTHORISED MATERIALS	EXAMINATION MATERIALS TO BE SUPPLIED
No additional printed material is permitted	1 x 20 Page Book 1 x Scrap Paper Formula Sheet/s

THIS EXAMINATION IS PRINTED DOUBLE-SIDED.

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Question 1

(a) (i) Write the order of the following ordinary differential equation (ODE):

$$y' + xe^{-x^2/2} = 0.$$ **[Marks 1]**

(ii) Solve the above ODE by integration. **[Marks 4]**

(iii) Show that $y = ce^{2x} - 2$ is a solution of the following ODE:

$$y' - 2y = 4\,.$$ **[Marks 3]**

(b) Solve the following separable ODE with the given initial condition:

$$y' = (x+1)e^{-x}y^2 \quad , y(0) = 1 \quad .$$ **[Marks 5]**

(c) Using a suitable substitution solve the following ODE:

$$xy' = y + 2x^3\sin^2(\frac{y}{x}).$$ **[Marks 7]**

Question 2

(a) Check if the following ODE is exact. If yes, then solve it.

$$\cos(x+y)\,dx + (3y^2 + 2y + \cos(x+y))dy = 0.$$ **[Marks 6]**

(b) Solve the following nonlinear ODE with the given initial condition:

$$y' + xy = xy^{-1}\,, y(0) = 3.$$ **[Marks 7]**

(c) In the study of motion of a small ball on a straight line it is found that the sum of its velocity and acceleration equals a constant K. Assuming that the distance travelled in time *t* seconds is $y(t)$) metres from the initial position y(0) and with the initial velocity v_0:

(i) Express this motion by a second order ODE. **[Marks 2]**

(ii) Find the solution of this ODE by first reducing it to first order.

[Marks 5]

Question 3

(a) Solve the following initial value problem:

$$y'' + y' - 6y = 0, \quad y(0) = 10, \quad y'(0) = 0.$$ **[Marks 6]**

(b) Given the following functions form the basis of solutions:

e^{-2x} and $e^{-x/2}$,

of the following differential equation:

$$y'' + ay' + by = 0.$$

Find the values of a and b. **[Marks 7]**

(c) Given that $y_1 = x^3$, find y_2 to form the basis of the following differential equation:

$$x^2 y'' - 5xy' + 9y = 0.$$ **[Marks 7]**

Question 4

(a) Find the general solution of the following nonhomogeneous ODE:

$$y'' + 3y' + 2y = 12e^{3x}.$$ **[Marks 7]**

(b) Find a general solution of the following system of linear differential equations:

$$\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= 3y_1 - y_2 \end{aligned} \quad .$$ **[Marks 7]**

(c) Identify the following curve written in the parametric representation and sketch it:

$$\mathbf{r}(\boldsymbol{t}) = [3 + 2\cos t, 2\sin t,\ t]$$ **[Marks 6]**

Question 5

(a) (i) Sketch the following periodic function $f(x)$ of period $p = 2\pi$.

$$f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } \quad 0 < x < \pi \end{cases}$$ **[Marks 5]**

(ii) Find the Fourier coefficients a_0, a_1 and b_1. **[Marks 9]**

(b) Find the length of the following curve:

$\mathbf{r}(t) = [6\cos t\,, 6\sin t]$ from (6,0) to (0,6). **[Marks 6]**

Question 6

(a) Find the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point

P : (2,1, 3) in the direction of vector $\boldsymbol{a} = [1, 0, -2]$. **[Marks 5]**

(b) Find the divergence of the following vector function:

$\mathbf{V} = 4x^2\mathbf{i} + 9y^2\mathbf{j} + z^2\mathbf{k}$ at point P: (5, -1, -11). **[Marks 5]**

(c) Find the line integral $\int_C \mathbf{F}(\mathbf{r}).d\mathbf{r}$ or $\int_a^b \mathbf{F}(\mathbf{r}).\frac{d\mathbf{r}}{dt}dt$ of $\mathbf{F} = xy\ \mathbf{i} + (y-x)^2\,\mathbf{j}$ on

C: $xy = 1,\ 1 \le x \le 3$. **[Marks 5]**

(d) Calculate the curl of the vector $\mathbf{v} = xyz[x, y, z]$. **[Marks 5]**

Faculty of EHSE
SMA209: Formula Sheet

- $\ln y = x \Leftrightarrow y = e^x$
- $\int \frac{dx}{x} = \ln x + c$
- $\int \sin x dx = -\cos x + c$
- $\int \cos x dx = \sin x + c$
- $\int e^{\alpha x} dx = \frac{1}{\alpha} e^{\alpha x} + c$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
- $\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c$
- $\int \sec^2 x dx = \tan x + c$
- $\int \csc^2 x dx = -\cot x + c$
- $\int \frac{dx}{\sinh^2 x} = -\coth x + c$
- $\int \frac{dx}{\cosh^2 x} = \tanh x + c$
- $\int \tan x dx = \ln|\sec x| + c$
- Integration by parts: $\int f(x)g(x)dx = f(x)\int g(x)dx - \int \ f'(x)(\int g(x)dx)dx$
- $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$
- Change of variable: $u = \frac{y}{x}$, $y = ux$ and $y' = u'x + u$
- Exact differential: $du = M(x,y)dx + N(x,y)dy = 0 \rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; $M = \frac{\partial u}{\partial x}, N = \frac{\partial u}{\partial y}$
- Not exact differential: For $Pdx + Qdy = 0$, $\frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}]$
- For $dy + (p(x)y - r(x))dx = 0$, $y = e^{-h}[\int e^h r(x)dx + c]$, $h = \int p(x)dx$
- To make, $y' + p(x)y = g(x)y^a$ linear, substitute $u = y^{1-a}$.
- Trial solution of: $y'' + ay' + by = 0$ is $y = e^{\lambda x}$ leading to $\lambda^2 + a\lambda + b = 0$.
- Distinct real roots; General solution $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
- For double roots: $y_2 = xy_1$,
- Complex roots: $y = e^{-\frac{a}{2}x}[A\cos\omega x + B\sin\omega x]$; $\omega = \frac{1}{2}\sqrt{4b - a^2}$
- Trial solution of: $x^2 y'' + axy' + by = 0$ is $y = x^m$ leading to $m^2 + (a-1)m + b = 0$.
- For double roots: $y_2 = y_1 \ln|x|$

- For a differential equation: $y'' + p(x)y' + q(x)y = 0$, if y_1 is known, then:

  $y_2 = y_1 \int U dx$ and $U = \frac{1}{y_1^2} e^{-\int p(x)dx}$

- $D = \frac{d}{dx}$
- $D^2 = \frac{d^2}{dx^2}$

- Wronskian, $W = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$

- For $y'' + p(x)y' + q(x)y = r(x)$, particular solution y_p may be constructed from the table below:

Table of undetermined coefficients

Term in r(x)	Choice for y(p)
ke^{ax}	Ce^{ax}
$kx^n (n = 0,1,...)$	$K_n x^n + K_{n-1}x^{n-1} + ... + K_1 x + K_0$
$\left.\begin{matrix} k\cos\omega x \\ k\sin\omega x \end{matrix}\right\}$	$K\cos\omega x + M\sin\omega x$
$\left.\begin{matrix} ke^{ax}\cos\omega x \\ ke^{ax}\sin\omega x \end{matrix}\right\}$	$e^{ax}(K\cos\omega x + M\sin\omega x)$

- In solving a system of linear equations:

$$\mathbf{y}' = \mathbf{A}\mathbf{x},$$

first determine the eigen values from $\det[\mathbf{A} - \lambda\mathbf{I}] = 0$, and then corresponding to each eigen value, solve:

$$[\mathbf{A} - \lambda\mathbf{I}]\mathbf{x} = 0.$$

- $f(x) = a_0 + \sum_n (a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x)$, period $2L$.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

- If *f*(*x*) is odd, then $\int_{-a}^{a} f(x)dx = 0$
- If *f*(*x*) is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$
- Tangent vector on curv C: $\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$
- $\mathbf{v} = \frac{d\mathbf{r}}{dt}$
- $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$
- $l = \int_a^b [\mathbf{r}'(t).\mathbf{r}'(t)]^{1/2} dt$
- $grad\ f = \nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\mathbf{k}$,
- $\nabla_a f = \mathbf{a}\cdot\nabla f$; **a** is unit vector.
- div $\mathbf{v} = \nabla.\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$
- Curl$\mathbf{v}$ = $\nabla\times\mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{bmatrix}$
- For $f(x, y.z) = 0$, normal $\mathbf{N} = \text{grad } f$, $\mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|}$.
- $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
- $\iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C (F_1 dx + F_2 dy)$
- $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$, $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$, $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$
- Flux: $\iint_S \mathbf{F}.\mathbf{n}dA = \iint \mathbf{F}[r(u,v)\}.\mathbf{N}(u,v)dudv$
- 2D Laplace's Eqn. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$