# SMA101-2019-Final examination guiding solutions

**Question 1**

(a) Solve $|x - 3| = 5$

This equation means

$$x - 3 = 5 \quad or \quad x - 3 = -5$$
$$x = 5 + 3 \quad or \quad x = -5 + 3$$
$$x = 8 \quad or \quad x = -2$$

Therefore

$$x = 8 \quad or \quad x = -2$$

(b)

i) $f(x) = \sqrt{x}$, $g(x) = x^3 + 1$

$f(g(x)) = \sqrt{g(x)}$

$= \sqrt{x^3 + 1}$

$f(g(2)) = \sqrt{2^3 + 1}$

$= \sqrt{9}$

$\therefore f(g(2)) = 3$

ii) $f(x) = \sqrt{x}$, $g(x) = x^3 + 1$

$g(f(x)) = f^3(x) + 1$

$= (\sqrt{x})^3 + 1$

$= x^{3/2} + 1$

$g(f(4)) = (\sqrt{4})^3 + 1$

$= 2^3 + 1$

$\therefore g(f(4)) = 9$

(c) $f(x) = \frac{1+x}{1-x}$

Writing it as: $y = \frac{1+x}{1-x}$, and then solving for $x$ gives:

$y = \frac{1+x}{1-x}$

$y(1-x) = x + 1$

$y - xy = x + 1$

$yx + x = y - 1$

$x(y + 1) = y - 1$

$\therefore x = \frac{y-1}{y+1} \therefore f^{-1}(x) = \frac{x-1}{x+1}$

(d) Find the limit of the following:

(i) Divide both numerator and denominator by $x^2$

$$\lim_{x\to-\infty} \frac{(\sqrt{3x^4+x})/x^2}{(x^2-8)/x^2}$$

$$\lim_{x\to-\infty} \frac{(\sqrt{3x^4/x^4+x/x^4})/x^2}{x^2/x^2-8/x^2}$$

$$\lim_{x\to-\infty} \frac{(\sqrt{3+1/x^3})}{1-8/x^2}$$

$$\frac{(\sqrt{3+0})}{1-0} = \sqrt{3}$$

Alternatively, As the limit is approaching$-\infty$, let $t = (-x)^2$

$$\lim_{x\to-\infty} \frac{\sqrt{3x^4+x}}{x^2-8}$$

$$\lim_{t\to-\infty} \frac{\sqrt{3t^2-\sqrt{t}}}{-t-8}$$

$$\lim_{t\to-\infty} \frac{\dfrac{(\sqrt{3t^2-\sqrt{t}})}{\sqrt{(-t)^2}}}{(\dfrac{-t-8}{-t})}$$

$$\lim_{t\to-\infty} \frac{\sqrt{\dfrac{3t^2}{t^2}-\dfrac{\sqrt{t}}{t^2}}}{(\dfrac{-t}{-t}-\dfrac{8}{-t})}$$

$$\lim_{t\to-\infty} \frac{\sqrt{3-\dfrac{1}{t^{3/2}}}}{(1+\dfrac{8}{-t})} = \frac{\sqrt{3-0}}{(1+0)} = \sqrt{3}$$

Or keep the highest power of *x* in both numerator and denominator

$$\lim_{x\to-\infty} \frac{\sqrt{3x^4}}{x^2} = \lim_{x\to-\infty} \frac{\sqrt{3}x^2}{x^2} = \sqrt{3}$$

(ii) $\lim_{t\to 2}\frac{t^3+3t^2-12t+4}{t^3-4t}$ At $x=2$ $\lim_{t\to 2}\frac{2^3+3(2)^2-12(2)+4}{2^3-4(2)}=\frac{0}{0}$

This is indeterminate hence we have to factorise both numerator and denominator.

To factorise the numerator put *t=2* in the numerator

$(2)^3+3(2)^2-12(2)+4=24-24=0$ hence 2 is a factor.

Do long division to obtain $t^3+3t^2-12t+4=(t^2+5t-2)(t-2)$

thus $\lim_{t\to 2}\frac{(t^2+5t-2)(t-2)}{t(t^2-4)} \Rightarrow \lim_{t\to 2}\frac{(t^2+5t-2)(t-2)}{t(t+2)(t-2)} \Rightarrow \lim_{t\to 2}\frac{(t^2+5t-2)}{t(t+2)}$

Now put *t=2* $\frac{(2^2+5(2)-2)}{2(2+2)}=\frac{12}{8}=\frac{3}{2}$

Alternatively, using the L'Hôpital's Rule, differentiate numerator and denominator

$$\lim_{t\to 2}\frac{3t^2+6t-12}{3t^2-4}$$

$$\frac{3(2^2)+6(2)-12}{3(2^2)-4}=\frac{12}{8}=\frac{3}{2}$$

**Question 2**

a)(i) $3+7x\leq 2x-9$ Add -3 each side

$\rightarrow 3+7x-3\leq 2x-9-3$

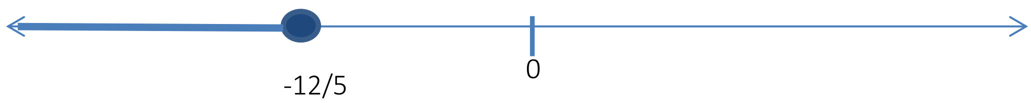
$\rightarrow 7x\leq 2x-12$ Add -2x each side

$\rightarrow 7x-2x\leq 2x-12-2x$

$\rightarrow 5x\leq -12$

$x\leq -\frac{12}{5}$

Solution is $(-\infty,-\frac{12}{5}]$



(ii) $\frac{2x-5}{x-2} < 1$ Note $x \neq 2$. Subtract 1 each side

$$\rightarrow \frac{2x-5}{x-2} - 1 < 1 - 1$$ Simplify the LHS

$$\rightarrow \frac{2x-5}{x-2} - 1\frac{(x-2)}{(x-2)} < 0$$

$$\rightarrow \frac{2x-5-x+2}{x-2} < 0$$

$$\rightarrow \frac{x-3}{x-2} < 0$$

The inequality is not satisfied at $x = 3$ *and* $x = 2$. Therefore the coordinate line has to be divided into 3 parts $(-\infty, 2)$, $(2, 3)$, $(3, +\infty)$

| Interval | Test point | Sign | comment |
|---|---|---|---|
| $(-\infty, 2)$ | 0 | (-)/(-)=+ | Not satisfied |
| $(2, 3)$ | 2.5 | (-)/(+)=- | Satisfied |
| $(3, +\infty)$ | 4 | (+)(+)=+ | Not satisfied |

Solution on a coordinate line :


(b) $s = 0.3t^3$ ft.

i) How high does the rocket travel in 40 s?

$$s = 0.3(40)^3 = 19200ft$$

ii) What is the average velocity of the rocket during the first 40 s?

$$= \frac{S_2-S_1}{t_2-t_1} = \frac{0.3(40)^3-0.3(0)^3}{40-0} = 480\ ft/s$$

iii) What is the average velocity of the rocket during the first 1000 ft of its flight?

The time to cover 1000 ft is found by writing $1000 = 0.3(t)^3$

$$t = \sqrt[3]{3333.33} = 14.93\ s$$

The average velocity corresponding to this time is

$$= \frac{S_2 - S_1}{t_2 - t_1} = \frac{0.3(14.93)^3 - 0.3(0)^3}{14.93 - 0} = 66.94\ ft/s$$

(iv) What is the instantaneous velocity of the rocket at the end of 40 s?

$$V_{inst} = \lim_{h\to 0} \frac{S(t+h) - S(t)}{h}$$

$$V_{inst} = \frac{ds}{dt} = \frac{d(0.3t^3)}{dt} = 0.3 \times 3 \times t^2$$

$$= 0.3 \times 3 \times (40)^2$$

$$= 1440\ ft/s$$

(c) i) $f(x) = 5x^4 - 3x + 7$

This is a polynomial, so no discontinuities. No value of x will make it undefined.

ii) $f(x) = \frac{x+2}{x^2-4}$

This can be written as:

$f(x) = \frac{x+2}{(x-2)(x+2)}$

The denominator vanishes at $x = 2$ and $x = -2$.

$\therefore$ it is discontinuous at $x = -2, 2$ because at these points the denominator is zero, thus undefined.

**Question 3**

a)

$$f(x) = \left[\ln\left(\frac{x^2 \sin x}{\sqrt{1+x^2}}\right)\right]$$

Expand the log function

$$f(x) = \ln x^2 + \ln \sin x - \ln\sqrt{1+x^2}$$
$$f(x) = 2\ln x + \ln \sin x - \frac{1}{2}\ln(1+x^2)$$

Take derivative of $f(x)$ w.r.t x

$$f'(x) = 2\frac{1}{x} + \frac{1}{\sin x}\cos x - \frac{1}{2}.\frac{1}{(1+x^2)}2x$$

$$f'(x) = \frac{2}{x} + \cot x - \frac{x}{(1+x^2)}$$

b) $\lim_{x\to 0} \frac{\sin cx}{x} = \frac{\sin c(0)}{0} = \frac{0}{0}$

Indeterminate of form 0/0 therefore L'Hopital's rule applies.

$$= \lim_{x\to 0} \frac{d(\sin cx)/dx}{d(x)/dx}$$

$$= \lim_{x\to 0} \frac{c \times \cos cx)}{1}$$

$$= \frac{c \times \cos c(0)}{1} = c$$

$$\therefore \lim_{x\to 0} \frac{\sin cx}{x} = c$$

c) Let $A$ be the area of the circular pattern and $r$ be the radius at any time $t$.

$$\frac{dA}{dt} = 6\ km^2/s$$

Area of a circle of radius r is

$$A = \pi r^2\ thus\ \frac{dA}{dr} = 2\pi r$$

Using chain rule

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{2\pi r} \frac{dA}{dt}$$

$$\frac{dr}{dt} = \frac{1}{2\pi r} \frac{dA}{dt}$$

Find the radius r from the area

$$A = 8 = \pi r^2$$

$$r = \sqrt{8/\pi}$$

$$\therefore \frac{dr}{dt} = \frac{1}{2\pi\sqrt{8/\pi}} 6 = \frac{3}{\sqrt{8\pi}}$$

$$\therefore \frac{dr}{dt} \approx 0.6\ km/s$$

**Question 4**

a) i) $y = \dfrac{(3x^2+2)}{x+3}$

$$\frac{dy}{dx} = \frac{(x+3)(6x)-(3x^2+2)}{(x+3)^2}$$

$= \dfrac{6x^2 + 18x - 3x^2 - 2}{(x+3)^2}$

$= \dfrac{3x^2 + 18x - 2}{(x+3)^2}$

ii) $f(x) = (2x^7 - x^2)\left(\dfrac{x-1}{x+1}\right)$

Applying the product rule:

$\dfrac{df}{dx} = (2x^7 - x^2)\,\dfrac{d}{dx}\left(\dfrac{x-1}{x+1}\right) + \left(\dfrac{x-1}{x+1}\right)\dfrac{d}{dx}(2x^7 - x^2)$

$= (2x^7 - x^2)\left[\dfrac{1\cdot(x+1)-(x-1)\cdot 1}{(x+1)^2}\right] + \left(\dfrac{x-1}{x+1}\right)(14x^6 - 2x)$

$= (2x^7 - x^2)\left[\dfrac{x+1-x+1}{(x+1)^2}\right] + \left(\dfrac{x-1}{x+1}\right)(14x^6 - 2x)$

$= (2x^7 - x^2)\left[\dfrac{2}{(x+1)^2}\right] + \left(\dfrac{x-1}{x+1}\right)(14x^6 - 2x)$

b) i) $\int_0^4 (x^3 - 4x - 3)\,dx = \left[\dfrac{x^4}{4} - \dfrac{4x^2}{2} - 3x\right]_0^4$

$= \left(\dfrac{4^4}{4} - 2\cdot 4^2 - 3\cdot 4\right) - (0)$

$= 64 - 32 - 12 \qquad = 20$

ii) $\int \dfrac{1}{1+\sin\theta}\,d\theta$

Multiply numerator and denominator by $1 - \sin\theta$

$= \int \dfrac{1-\sin\theta}{1-\sin^2\theta}\,d\theta \qquad\qquad 1 - \sin^2\theta = \cos^2\theta$

$= \int \dfrac{1-\sin\theta}{\cos^2\theta}\,d\theta$

$= \int \dfrac{1}{\cos^2\theta} - \dfrac{\sin\theta}{\cos^2\theta}\,d\theta$

$= \int \sec^2\theta\,d\theta - \int \tan\theta\,\dfrac{1}{\cos\theta}\,d\theta$

$= \int \sec^2\theta\,d\theta - \int \tan\theta\sec\theta\,d\theta \qquad\qquad \int \sec^2\theta d\theta = \tan\theta + c$

$= \tan\theta - \sec\theta + c$ $\qquad\qquad$ $\int \tan\theta \sec\theta d\theta = \sec\theta + c$

(c) $\quad \det(A) = \begin{vmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{vmatrix} = (\lambda-4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda-1 \end{vmatrix}$

$= (\lambda-4)[(\lambda)(\lambda-1) - 6]$

$= (\lambda-4)(\lambda^2-\lambda-6)$

for det($A$)= 0

$(\lambda-4)(\lambda^2-\lambda-6) = 0$

$\therefore \lambda = 4$ $\qquad$ one value

$(\lambda^2-\lambda-6) = 0$ $\qquad$ factorise $\qquad$ $(\lambda-3)(\lambda+2) = 0$

$\therefore \lambda = 3$ or $\lambda = -2$

Possible values are $\lambda_1 = 4,\ \lambda_2 = -2,\ \lambda_3 = 3$

**Question 5**

a) Norm of vector **w** = (−7, 2, −1).

$$\|\mathbf{w}\| = \sqrt{(-7)^2 + (2)^2 + (-1)^2} = \sqrt{54} = 3\sqrt{6}$$

b)

$$[A \mid I] = \left[\begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right]$$

Multiply rows 1, 2 and 3 by 5 to get

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & 0 & 5 & 0 \\ 1 & -4 & \frac{1}{2} & 0 & 0 & 5 \end{array}\right]$$

Add −1 × row 3 to row 2

$$[A\,|\,I\,] = \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 5 & 0 & 0 & 5 & -5 \\ 1 & -4 & \frac{1}{2} & 0 & 0 & 5 \end{array}\right]$$

Add $-1 \times$ row 1 to row 3

$$[A\,|\,I\,] = \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 5 & 0 & 0 & 5 & -5 \\ 0 & -5 & 2\frac{1}{2} & -5 & 0 & 5 \end{array}\right]$$

Divide row 2 and row 3 by 5

$$[A\,|\,I\,] = \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & \frac{1}{2} & -1 & 0 & 1 \end{array}\right]$$

Add row 2 to row 3

$$[A\,|\,I\,] = \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & -1 & 1 & 0 \end{array}\right]$$

Multiply row 3 by 2 to get

$$[A\,|\,I\,] = \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right]$$

Add $2 \times$ row 3 to row 1

$$[A\,|\,I\,] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right]$$

Add $-1 \times$ row 2 to row 1

$$[A\,|\,I\,] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}$$

c) Find the minor $M_{22}$ and cofactor $C_{22}$ of the matrix:

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

Since minor is the determinant of a submatrix, the determinant of the submatrix obtained by removing second row and second column is

$$M_{22} = \begin{vmatrix} 4 & 1 & 6 \\ 4 & 0 & 14 \\ 4 & 3 & 2 \end{vmatrix}$$

Expansion along the first column give

$$M_{22} = 4\begin{vmatrix} 0 & 14 \\ 3 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 6 \\ 0 & 14 \end{vmatrix}$$

$$M_{22} = 4(-42) - 4(-16) + 4(14)$$

$$M_{22} = -168 + 64 + 56 = -48$$

The cofactor $C_{22} = (-1)^{2+2}M_{22} = 1(-48) = -48$

(d) **u** = (−6, 4, 2), **v** = ( 3, 1, 5)

The vector that is perpendicular to both **u** and **v** is **u** × **v**

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 4 & 2 \\ 3 & 1 & 5 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -6 & 2 \\ 3 & 5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -6 & 4 \\ 3 & 1 \end{vmatrix}$$

$= \mathbf{i}\,(20 - 2) - \mathbf{j}\,(-30 - 6) + \mathbf{k}\,(-6 - 12)$

$= 18\mathbf{i} + 36\mathbf{j} - 18\mathbf{k}$

$= (18, 36, -18)$

**Question 6**

a) **u** = (3, 4), **v** = (5, −1), **w** = (7, 1)

i) **u • (7v + w)**

7**v** = (35, −7)

7**v** + **w** = (35, −7) + (7, 1)

= (42, −6)

**u** • (7**v** + **w**) = (3, 4) • (42, −6)

= 126 − 24

$= 102$

ii) $\|\mathbf{u}\|(\mathbf{v} \bullet \mathbf{w})$

$\|\mathbf{u}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\mathbf{v} \bullet \mathbf{w} = 35 - 1$

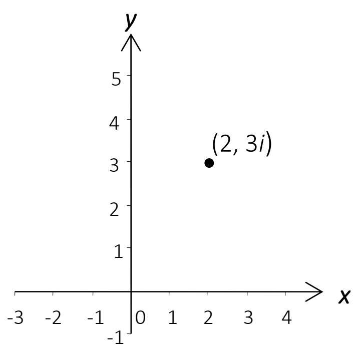
$= 34$

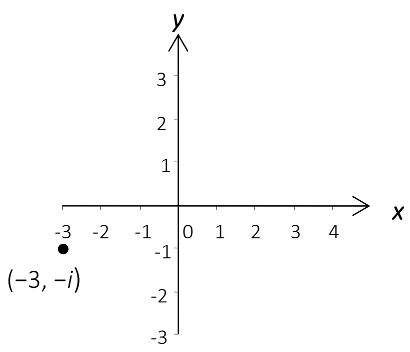
$\|\mathbf{u}\|(\mathbf{v} \bullet \mathbf{w}) = 5 \times 34$

$= 170$

(b) Complex numbers shown as **points** on an Argand diagram.

i) $2 + 3i$ ii) $-3 - i$




(c) $z = (1 + \sqrt{3}i)^{1/2}$

$1 + \sqrt{3}i = r(\cos\theta + i\sin\theta)$

$r = \left|1 + \sqrt{3}i\right| = \sqrt{1+3} = 2$

$\therefore\ 1 + \sqrt{3}i = 2(\cos\theta + i\sin\theta)$

$\therefore\ \cos\theta + i\sin\theta = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

This gives: $\cos\theta = \dfrac{1}{2},\ \sin\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = \dfrac{\pi}{3}$

$\therefore 1 + \sqrt{3}i = 2(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3})$

$$\Rightarrow (1+\sqrt{3}i)^{1/2} = 2^{1/2}(\cos\left(\frac{\pi}{6}+\frac{2k\pi}{2}\right) + i\sin\left(\frac{\pi}{6}+\frac{2k\pi}{2}\right)), \quad k = 0, 1$$

First root $k = 0$

$$= 2^{1/2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \sin\frac{\pi}{6} = \frac{1}{2}$$

$$= \sqrt{2}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{2}+i\frac{\sqrt{2}}{2}$$

Second root $k = 1$

$$2^{1/2}(\cos\left(\frac{\pi}{6}+\frac{2\pi}{2}\right) + i\sin\left(\frac{\pi}{6}+\frac{2\pi}{2}\right))$$

$$= 2^{1/2}(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})$$

$$= 2^{1/2}(\cos\left(\pi+\frac{\pi}{6}\right) + i\sin\left(\pi+\frac{\pi}{6}\right))$$

$$= 2^{1/2}(-\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right))$$

$$= \sqrt{2}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)$$

$$= -\frac{\sqrt{6}}{2}-i\frac{\sqrt{2}}{2}$$

(d) $i(1 + 7i) - 3i\,(4 + 2i)$

$$= i + 7i^2 - 12i - 6i^2$$

$$= i - 7 - 12i + 6$$

$$= -1 - 11i$$