Extremal Problems
in
Finite Sets

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I hereby declare that the work herein, now submitted as a thesis for the degree of Doctor of Philosophy, is the result of my own investigations, and all references to ideas and work of other researchers have been specifically acknowledged. I hereby certify that the work embodied in this thesis has not already been accepted in substance for any degree, and is not being currently submitted in candidature for any other degree.

(Dated & Signed)____________________________________

Paulette Lieby
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Abstract

The main concern of this thesis is the study of problems involving collections of subsets of a finite set $[n] = \{1, 2, \ldots, n\}$ and especially antichains on $[n]$. An antichain is a collection of sets in which no two sets are comparable under set inclusion. An antichain $\mathcal{A}$ is flat if there exists an integer $k \geq 0$ such that every set in $\mathcal{A}$ has cardinality either $k$ or $k + 1$. The size of $\mathcal{A}$ is $|\mathcal{A}|$ and the volume of $\mathcal{A}$ is $\sum_{A \in \mathcal{A}} |A|$. A unifying problem in the thesis is the flat antichain conjecture which states that $\mathcal{A}$ is an antichain on $[n]$ if and only if there exists a flat antichain on $[n]$ with the same size and volume as $\mathcal{A}$. The truth of the conjecture would provide a simple test for the existence of antichains on $[n]$ with a given size and volume. The flat antichain conjecture is known to hold in several special cases. This thesis shows that it holds in several further cases.

Two main approaches have been taken in the investigation of the flat antichain conjecture. The first approach consists of studying the volumes of antichains. Building on earlier work of Clements we observe that for given $n$ and $s$, the antichains on $[n]$ of size $s$ which achieve minimum (maximum) volume are flat antichains, where the minimum (maximum) is taken over all antichains on $[n]$ of size $s$. Further, we show that if $\mathcal{A}$ is an antichain on $[n]$ then there exists a flat antichain on $[n]$ with the same volume as $\mathcal{A}$. In proving this last result we prove that the flat antichain conjecture holds in one special case.

The second approach involves counting the number of subsets and supersets of certain collections of sets as defined below. The squashed (or colex) order on sets is a set ordering with the property that the number of subsets of a collection of $k$-sets is minimised when the collection consists of an initial segment of $k$-sets in squashed order. In the universal set $[n]$, consider the collections $L_{k+1}(p)$ and $L_{k-1}(p)$ consisting of the last $p$ ($k + 1$)-sets in squashed order and the last $p$ ($k - 1$)-sets in squashed order respectively. Let $a_1$ be the number of supersets of size $k$ of the sets in $L_{k-1}(p)$. Let $a_2$ be the number of subsets of size $k$ of the sets in $L_{k+1}(p)$ which are not subsets of any ($k + 1$)-set preceding the sets in $L_{k+1}(p)$ in squashed order. We show that $a_1 + a_2 > 2p$. This result, called the 3-levels result, enables us to show that the flat antichain conjecture holds in several cases.

Several conjectures and open problems which are related to the 3-levels result are stated and implications of the truth of the flat antichain conjecture are considered.
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Chapter 1

Introduction
1.1 Overview

This thesis is concerned with the study of collections of subsets of a finite set \([n] = \{1, 2, \ldots, n\}\) and especially antichains on \([n]\). An antichain \(\mathcal{A}\) on \([n]\) is a collection of subsets of \([n]\) such that no member of \(\mathcal{A}\) is a proper subset of any other member of \(\mathcal{A}\).

The starting point of the theory of antichains (or Sperner theory) is Sperner’s theorem (Theorem 2.74) which determines the maximum number of sets in an antichain on \([n]\). Two further important results are the Kruskal-Katona theorem (Theorem 2.36) and an existence theorem (Theorem 2.77) independently discovered by Clements and Daykin et al. The Kruskal-Katona theorem shows that the number of subsets of a collection of sets with same cardinality is minimised when the collection is taken to consist of an initial segment of a specific ordering of sets. This ordering is called the squashed order (or colex order) and it orders the sets in such a way as to use as few elements as possible. The existence theorem, which is a consequence of the Kruskal-Katona theorem, states necessary and sufficient conditions for a given antichain to exist.

Let \(\mathcal{B}\) be a collection of subsets of \([n]\). The size (or cardinality) \(|\mathcal{B}|\) of \(\mathcal{B}\) is the number of sets in \(\mathcal{B}\) and the volume of \(\mathcal{B}\) is \(V(\mathcal{B}) = \sum_{B \in \mathcal{B}} |B|\), the sum of the cardinalities of the sets of \(\mathcal{B}\). The parameters of \(\mathcal{B}\) are the integers \(p_i, 0 \leq i \leq n\), which denote the number of sets of cardinality \(i\), or \(i\)-sets, in \(\mathcal{B}\). The profile of \(\mathcal{B}\) is the \((n+1)\)-tuple \((p_0, p_1, \ldots, p_n)\).

One problem in Sperner theory is to test if there exists an antichain of subsets of \([n]\) which has certain characteristics. The existence theorem mentioned above states necessary and sufficient conditions for non-negative integers \(p_0, p_1, \ldots, p_n\) to be the parameters of an antichain on \([n]\). In some cases however, it may be desired to determine if an antichain exists where the characteristics of the candidate antichain are not given in terms of the parameters of a collection of sets. For example, the problem may be to test if an antichain on \([n]\) exists with size \(s\) and volume \(V\). Given \(s\) and \(V\) it is possible to find all possible profiles \((p_0, p_1, \ldots, p_n)\) such that \(\sum_{i=0}^{n} p_i = s\) and \(\sum_{i=0}^{n} ip_i = V\). Applying the existence theorem to each such profile
enables one to decide if an antichain on $[n]$ with size $s$ and volume $V$ exists. However, finding all possible profiles satisfying the conditions $\sum_{i=0}^{n} p_i = s$ and $\sum_{i=0}^{n} i p_i = V$ is computationally expensive.

The flat antichain conjecture (Conjecture 4.1) arose from the consideration of this problem. The flat antichain conjecture says that if $\mathcal{A}$ is an antichain on $[n]$ then there exists an antichain on $[n]$ with the same size and volume as $\mathcal{A}$ and with profile $(p_0, p_1, \ldots, p_n)$ where $p_i = 0$ for $i \neq k$, $k + 1$ for some $k$, $0 \leq k < n$. An antichain, or more generally a collection of sets, which has such a profile is called a flat antichain, or a flat collection of sets respectively. Therefore, an equivalent formulation of the flat antichain conjecture is that $\mathcal{A}$ is an antichain on $[n]$ if and only if there exists a flat antichain $\mathcal{A}^*$ on $[n]$ with $|\mathcal{A}^*| = |\mathcal{A}|$ and $V(\mathcal{A}^*) = V(\mathcal{A})$.

Let $n$, $s$ and $V$ be given and assume that the flat antichain conjecture holds. It is not difficult to determine the parameters $p_i$ of a flat collection of subsets of $[n]$ with size $s$ and volume $V$. Then, to determine if there exists an antichain on $[n]$ with size $s$ and volume $V$, it is sufficient to test, by applying the existence theorem, if the $p_i$'s are the parameters of an antichain on $[n]$. For if the $p_i$'s are not the parameters of an antichain on $[n]$, then, assuming that the flat antichain conjecture holds, there exists no antichain on $[n]$ with size $s$ and volume $V$.

The flat antichain conjecture is a new kind of extremal problem about antichains which specifically involves the consideration of the volumes of antichains rather than just their sizes. The flat antichain conjecture and the approaches taken to solve it form the main thread in the thesis. To date the conjecture is still open.

The main definitions and notations are introduced in Section 1.2 of this chapter. Chapter 2 is a survey of some results in the combinatorics of finite sets and provides the background material required in the later chapters. Most results in Chapter 2 are standard results from the literature, although some are new and these are marked by $\triangledown$. Chapter 3 presents and proves improved versions of two theorems by Clements (Theorems 2.47 and 2.48) which are needed in establishing results in Chapter 8.

Chapter 4 presents the flat antichain conjecture and the original motivation which led
Chapter 1. Introduction

to its formulation. Several cases where the conjecture is known to hold are given. In this same chapter it is also shown that if \( \mathcal{A} \) is an antichain on \([n]\), then the parameters of a flat collection of subsets of \([n]\) with the same size and volume as \( \mathcal{A} \) satisfy the LYM inequality (Theorem 2.75). The LYM inequality is a necessary condition for integers to be the parameters of an antichain on \([n]\).

Chapters 5 and 6 investigate the volumes of antichains. In Chapter 5 some results drawn from the literature are discussed. In particular, we note that it follows from the work of Clements that for given \( n \) and \( s \), any antichain on \([n]\) and of size \( s \) which achieves minimum (maximum) volume is a flat antichain, where the minimum (maximum) is taken over all antichains on \([n]\) of size \( s \). Chapter 6 proves that a weaker version of the flat antichain conjecture holds. It is shown that given any antichain \( \mathcal{A} \) on \([n]\), there exists a flat antichain \( \mathcal{A}^* \) on \([n]\) with volume \( V(\mathcal{A}^*) = V(\mathcal{A}) \). Note that there is no requirement regarding the size of \( \mathcal{A}^* \). Theorem 6.7 shows that the flat antichain conjecture holds in one special case.

Chapter 7 takes an approach which consists of counting the number of subsets and supersets of certain collections of sets. Denote the collection of the last \( p \) \((k+1)\)-subsets of \([n]\) in squashed order by \( L_{k+1}(p) \) and the collection of the last \( p \) \((k-1)\)-subsets of \([n]\) in squashed order by \( L_{k-1}(p) \). Let \( a_1 \) be the number of \( k \)-sets which are supersets of the sets in \( L_{k-1}(p) \). Let \( a_2 \) be the number of \( k \)-sets which are subsets of the sets in \( L_{k+1}(p) \) but which are not subsets of any \((k+1)\)-set preceding the sets in \( L_{k+1}(p) \) in squashed order. The 3-levels result (Theorem 7.1) shows that \( a_1 + a_2 > 2p \) whenever \( p \) is non-zero. A long and complex proof is needed to prove the 3-levels result. This is the object of Chapter 7.

Chapter 8 derives some consequences of the 3-levels result. It is shown that the flat antichain conjecture holds in several cases as given by Theorems 8.2, 8.7, 8.14 and 8.19, and Corollaries 8.4 and 8.5. In particular, the flat antichain conjecture holds for antichains having sets on three or four consecutive levels, where the levels designate the cardinalities of the sets in the antichain. Chapter 8 also presents one generalisation of the 3-levels result.
Chapter 9 discusses further conjectures and open problems, most of them involving the number of subsets and the number of supersets of appropriately chosen collections of sets. These conjectures and open problems arose while proving the 3-levels result and while attempting to prove that the flat antichain conjecture holds by using the 3-levels result.

Chapter 10 explores some of the consequences of the flat antichain conjecture if one assume that it holds. In particular, necessary and sufficient conditions for the existence of antichains on \([n]\) with given size and volume are stated.

New results are marked by \(\Diamond\) in Chapter 2 only. In the remaining chapters any result without a reference is a new result.

1.2 Definitions and Notation

This section introduces most of the definitions and symbols which are used throughout the thesis. A list of symbols and definitions is given in Tables A.1 and B.1 which are found in Appendices A and B respectively. Most of the definitions and terminology are standard. Subsection 1.2.1 introduces sets and ordering on sets, Subsection 1.2.2 defines shadows and shades of a collection of sets, Subsection 1.2.3 discusses antichains, and Subsection 1.2.4 is a collection of miscellaneous definitions needed throughout the thesis.

1.2.1 Sets, Collections of Sets, and Orderings on Sets

Throughout the thesis the universal set is the finite set \(\{1, \ldots, n\}\) which is denoted by \([n]\). The size or cardinality of a set \(B\) is the number of elements in the set and is denoted by \(|B|\). If \(|B| = k\), then \(B\) is a \(k\)-set or a \(k\)-subset. Alternatively we say that \(B\) is a set on level \(k\). The collection of all the \(k\)-subsets of \([n]\) is denoted by \([n]^k\).

For sets \(A\) and \(B\), the set difference of \(A\) and \(B\) is \(A \setminus B = \{i : i \in A, i \notin B\}\). The
The symmetric difference of \( A \) and \( B \) is \( A \triangle B = (A \setminus B) \cup (B \setminus A) \). The complement of a subset \( B \) of \([n]\) is \( B' = [n] \setminus B \).

Let \( \mathcal{B} \) be a collection of subsets of \([n]\). The size or cardinality of \( \mathcal{B} \) is the number of sets in \( \mathcal{B} \) and is denoted by \(|\mathcal{B}|\). The volume of \( \mathcal{B} \) is \( V(\mathcal{B}) = \sum_{B \in \mathcal{B}} |B| \) and its average set size is \( \overline{|B|} = \frac{V(\mathcal{B})}{|\mathcal{B}|} \). The complement of \( \mathcal{B} \) is \( \mathcal{B}' = \{ B : B' \in \mathcal{B} \} \). If the sets in \( \mathcal{B} = \{ B_1, B_2, \ldots, B_m \} \) are ordered, then it is assumed that \( \mathcal{B}' = \{ B'_1, B'_2, \ldots, B'_m \} \). The collection of the \( i \)-sets in \( \mathcal{B} \) is denoted by \( \mathcal{B}^{(i)} = \{ B : B \in \mathcal{B}, |B| = i \} \). The parameters of \( \mathcal{B} \) are the integers \( p_i = |\mathcal{B}^{(i)}|, 0 \leq i \leq n \), and its levels are the integers \( i \). The profile of \( \mathcal{B} \) is the \((n+1)\)-tuple \((p_0, \ldots, p_n)\).

The collection \( \mathcal{B} \) is flat if for all \( B \in \mathcal{B}, |B| = |\overline{B}| \) or \(|B| = |\overline{B}| + 1 \). That is, \( \mathcal{B} \) is flat if it consists of sets on at most two consecutive levels. We say that the collection of sets \( \mathcal{B}^* \) is a flat counterpart of \( \mathcal{B} \) if \( \mathcal{B}^* \) is flat with \(|\mathcal{B}^*| = |\mathcal{B}| \) and \( V(\mathcal{B}^*) = V(\mathcal{B}) \).

When no ambiguity arises the braces are left out when writing sets: The set \( \{a, b, c\} \) may be written \( abc \).

Example 1.1. Let \( \mathcal{B} = \{1234, 123, 124, 15, 25, 6\} \). Then \( \mathcal{B}^* = \{123, 124, 134, 15, 25, 35\} \) is a flat counterpart of \( \mathcal{B} \) since \(|\mathcal{B}^*| = |\mathcal{B}| = 6 \) and \( V(\mathcal{B}^*) = V(\mathcal{B}) = 15 \).

The ideal of \( \mathcal{B} \) is the collection \( \mathcal{I}\mathcal{B} = \{ D : D \subseteq B, B \in \mathcal{B} \} \) of the subsets of the sets in \( \mathcal{B} \). The filter of \( \mathcal{B} \) is the collection \( \mathcal{F}\mathcal{B} = \{ D \subseteq [n] : D \supseteq B, B \in \mathcal{B} \} \) of the supersets in \([n]\) of the sets in \( \mathcal{B} \). The ideal on level \( k \), \( 0 \leq k \leq n \), is \( \mathcal{I}^{(k)}\mathcal{B} = \{ D : D \in \mathcal{I}\mathcal{B}, |D| = k \} \). The filter on level \( k \), \( 0 \leq k \leq n \), is \( \mathcal{F}^{(k)}\mathcal{B} = \{ D : D \in \mathcal{F}\mathcal{B}, |D| = k \} \).

Example 1.2. Let \( \mathcal{B} = \{123, 14, 5\} \). Then \( \mathcal{I}\mathcal{B} = \{123, 12, 13, 23, 14, 1, 2, 3, 4, 5, \emptyset\} \) and \( \mathcal{I}^{(2)}\mathcal{B} = \{12, 13, 23, 14\} \). For \( n = 5 \), \( \mathcal{F}^{(i)}\mathcal{B} = \{1234, 1235, 1245, 1345, 2345\} \).

A partition of \( \mathcal{B} \) is a collection of pairwise disjoint subcollections of \( \mathcal{B} \) whose union is \( \mathcal{B} \). That is, the collection \( \pi_1 = \{ B_1, B_2, \ldots, B_m \} \) with \( B_i \cap B_j = \emptyset \), \( 1 \leq i < j \leq m \), and \( \bigcup_{i=1}^{m} B_i = \mathcal{B} \) is a partition of \( \mathcal{B} \). Note that in this definition of a partition of \( \mathcal{B} \), the subcollections are allowed to be empty.
Let $L$ be a set such that $b < l$ for all $b \in B \in \mathcal{B}$ and all $l \in L$. Then $\mathcal{B} \uplus L$ is defined to be $\mathcal{B} \uplus L = \{ D : D = B \cup L, B \in \mathcal{B} \}$.

**Example 1.3.** Let $\mathcal{B} = \{1, 13, 23\}$ and $L = \{56\}$. Then $\mathcal{B} \uplus L = \{156, 1356, 2356\}$.

An order relation on sets, the **squashed order**, denoted by $\leq_S$, is defined by: If $A$ and $B$ are any sets, then $A \leq_S B$ if the largest element in $A + B$ is in $B$ or if $A = B$. We write $A <_S B$ or $B >_S A$ if $A \leq_S B$ and $A \neq B$. Let $\mathcal{B}$ and $\mathcal{C}$ be two collections of sets in squashed order. We write $A <_S B$ or $B >_S A$ if $A <_S B$ for all $B \in \mathcal{B}$, and $A >_S B$ or $B <_S A$ if $A >_S B$ for all $B \in \mathcal{B}$. We write $\mathcal{B} <_S \mathcal{C}$ or $\mathcal{C} >_S \mathcal{B}$ if $B <_S C$ for all $B \in \mathcal{B}$ and $C \in \mathcal{C}$.

The reverse of the squashed order for subsets of $[n]$ is called the **antilexicographic order** and is denoted by $\leq_A$. That is, $A \leq_A B$ implies that the largest element of $A + B$ is in $A$ or $A = B$.

**Example 1.4.** The first ten 3-sets in squashed order are: 123, 124, 134, 234, 125, 135, 235, 145, 245, 345.

We see that, in particular, $123 <_S \{124, 124\}$ and $\{125, 135\} <_S \{145, 245, 345\}$.

Let $\mathcal{B}$ be a collection of $p$ sets in squashed order. If $\mathcal{B} = F_{n,k}(p)$ we say that $\mathcal{B}$ is an \textit{initial segment} of $k$-sets \textit{in squashed order} or that $\mathcal{B}$ is a \textit{terminal segment} of $k$-sets \textit{in antilexicographic order}. If $\mathcal{B} = L_{n,k}(p)$ we say that $\mathcal{B}$ is a \textit{terminal segment} of $k$-subsets of $[n]$ \textit{in squashed order} or that $\mathcal{B}$ is an \textit{initial segment} of $k$-subsets of $[n]$ \textit{in antilexicographic order}. Finally, $F(p, \mathcal{B})$ and $L(p, \mathcal{B})$ respectively denote the first and the last $p$ sets of $\mathcal{B}$ in squashed order.

**Example 1.5.** Using Example 1.4, we see that $F_{5,3}(2) = \{123, 124\}$, $L_{5,3}(2) = \{245, 345\}$ and that the collection $\{125, 135\}$ is a collection $C_{5,3}(2)$. Also, $N_{5,3}(2) = \{134, 234\}$ and $P_{5,3}(2) = \{235, 145\}$.

Let $\mathcal{B} = \{234, 125, 135\}$. Then $F(2, \mathcal{B}) = \{234, 125\}$ and $L(2, \mathcal{B}) = \{125, 135\}$.

### 1.2.2 Shadows and Shades

Let $B$ be a $k$-subset of $[n]$. The \textit{shadow} of $B$ is $\triangle B = \{D : D \subset B, |D| = k - 1\}$ and its \textit{shade} is $\nabla B = \{D \subseteq [n] : D \supset B, |D| = k + 1\}$. The \textit{new-shadow} of $B$ is $\triangle_N B = \{D : D \in \triangle B, D \notin \triangle C \text{ for all } C \prec_S B\}$. That is, $\triangle_N B$ is the collection of the $(k - 1)$-sets which belong to the shadow of $B$ but \textit{not} to the shadow of any $k$-set which precedes $B$ in squashed order. In other words, the new-shadow of $B$ can be thought of as being the extra contribution of $B$ to the shadow of the first $p$ $k$-sets in squashed order, where $B$ is the $p$th set in squashed order. Similarly, the \textit{new-shade} of $B$ is $\nabla_N B = \{D : D \in \nabla B, D \notin \nabla C \text{ for all } C \succ_S B\}$. That is, $\nabla_N B$ consists of the $(k + 1)$-sets which are in the shade of $B$ but \textit{not} in the shade of any $k$-set which follows $B$ in squashed order.

Let $\mathcal{B}$ be a collection of $k$-subsets of $[n]$. The \textit{shadow} of $\mathcal{B}$ is $\triangle \mathcal{B} = \bigcup_{B \in \mathcal{B}} \triangle B$ and its \textit{shade} is $\nabla \mathcal{B} = \bigcup_{B \in \mathcal{B}} \nabla B$. The \textit{new-shadow} of $\mathcal{B}$ is $\triangle_N \mathcal{B} = \bigcup_{B \in \mathcal{B}} \triangle_N B$ and its \textit{new-shade} is $\nabla_N \mathcal{B} = \bigcup_{B \in \mathcal{B}} \nabla_N B$.

**Example 1.6.** Let $[n] = [5]$. For each 3-subset of $[5]$, we list the sets in its new-shadow and the sets in its new-shade. The 3-sets are listed in squashed order.
Let \( B \) and \( C \) be collections of sets such that \(|B| = |C|\). We say that \( B \) corresponds to \( C \) (and vice versa) if \(|\Delta_N B| = |\Delta_N C|\) and \(|\nabla_N B| = |\nabla_N C|\). Alternatively, we say that \( B \) and \( C \) correspond or that there is a correspondence between \( B \) and \( C \).

**Example 1.7.** Let \([n] = [5]\) and let \( B = \{125, 135, 235\} \) and \( C = \{12, 13, 23\}\). From Example 1.6 we see that \( \Delta_N B = \{15, 25, 35\} \) and \( \nabla_N B = \{1235\} \). Also, \( \Delta_N C = \Delta_N F_{5,2}(3) = \{1, 2, 3\} \) and \( \nabla_N C = \{123\} \) since any set \( B \in \nabla C \), \( B \neq 123 \), contains an element larger than 3 and is thus a superset of a 2-set following \( C \) in squashed order. It follows that \( B \) and \( C \) correspond.

Let \( B \subseteq [n] \). The **shadow on level** \( l \) of \( B \) is \( \Delta^{(l)} B = \{D : D \subseteq B, |D| = l\} \) and its **shade on level** \( l \) is \( \nabla^{(l)} B = \{D \subseteq [n] : D \supset B, |D| = l\} \). The **new-shadow on level** \( l \) of \( B \) is \( \Delta_N^{(l)} B = \{D : D \in \Delta^{(l)} B, D \not\in \Delta^{(l)} C \text{ for all } C <_S B\} \) and its **new-shade on level** \( l \) is \( \nabla_N^{(l)} B = \{D : D \in \nabla^{(l)} B, D \not\in \nabla^{(l)} C \text{ for all } C >_S B\} \).
Let $B$ be a collection of subsets of $[n]$, not necessarily all of size $k$. The shadow on level $l$ of $B$ is $\Delta^{(i)} B = \bigcup_{B \in B} \Delta^{(i)} B$ and its shade on level $l$ is $\nabla^{(i)} B = \bigcup_{B \in B} \nabla^{(i)} B$. The new-shadow on level $l$ of $B$ is $\Delta^{(i)}_N B = \bigcup_{B \in B} \Delta^{(i)}_N B$ and its new-shade on level $l$ is $\nabla^{(i)}_N B = \bigcup_{B \in B} \nabla^{(i)}_N B$.

**Example 1.8.** Let $[n] = [6]$ and $B = \{1236, 1246, 156\}$. Then $\Delta^{(3)} B = \{123, 124, 126, 136, 236, 146, 246\}$ and $\Delta^{(3)}_N B = \{126, 136, 236, 146, 246\}$. Note that $\{156\} \not\subseteq \Delta^{(3)} B$ so $\Delta^{(3)} B \neq \mathcal{I}^{(3)} B$.

Also, $\nabla^{(4)} B = \{1256, 1356, 1456\}$ and $\nabla^{(4)}_N B = \emptyset$. Note that $\{1236, 1246\} \not\subseteq \nabla^{(4)} B$ so $\nabla^{(4)} B \neq \mathcal{F}^{(4)} B$.

The $l$-projection of a collection of sets $B$ is $\diamond^{(i)} B = \bigcup \{B^l \subseteq B \}$ and its $l$-image is $\diamond^{(i)} B = \nabla^{(i)} B$.

**Example 1.9.** Let $B = \{1234, 125, 135, 45, 16\}$. Then $\diamond^{(3)} B = \{123, 124, 134, 234, 125, 135, 145, 245, 345, 126, 136, 146, 156\}$ and $\diamond^{(3)} B = \{123, 124, 134, 234, 125, 135, 145, 245, 345\}$. Observe that $\diamond^{(2)} B = \diamond^{(2)} B = \mathcal{I}^{(2)} B = \{12, 13, 23, 14, 24, 34, 15, 25, 35, 45, 16\}$ is a collection of consecutive 2-sets in squashed order but that $\diamond^{(3)} B$ and $\diamond^{(3)} B$ do not consist of consecutive 3-sets in squashed order.

### 1.2.3 Antichains

An antichain on $[n]$ is a collection of subsets of $[n]$ such that no two sets in the collection are comparable under set inclusion. That is, $\mathcal{A}$ is an antichain if for any $A, B \in \mathcal{A}$, $A \not\subseteq B$. Let $\mathcal{A}$ be an antichain on $[n]$ with largest and smallest set size $h$ and $l$ respectively. The antichain $\mathcal{A}$ is *squashed* if, for $i = h, h - 1, \ldots, l$, the $i$-sets in $\mathcal{A}^{(i)}$ precede the sets in $\mathcal{A}^{(i)}$ in squashed order so that the sets in $\mathcal{F}^{(i)} \mathcal{A} = \Delta^{(i)} \mathcal{A} \cup \mathcal{A}^{(i)}$ form an initial segment of the $i$-sets in squashed order. The antichain $\mathcal{A}$ is *antilexicographic* if, for $i = l, l + 1, \ldots, h$, the $i$-sets in $\nabla^{(i)} \mathcal{A}$ follow the sets in $\mathcal{A}^{(i)}$ in squashed order so that the sets in $\mathcal{F}^{(i)} \mathcal{A} = \mathcal{A}^{(i)} \cup \nabla^{(i)} \mathcal{A}$ form an initial segment of the $i$-subsets of $[n]$ in antilexicographic order.

Let $\mathcal{A}$ be a squashed antichain on $[n]$ with largest and smallest set size $h$ and $l$ respectively. The antichain $\mathcal{A}$ is *reducible* if there exists $t \in \mathbb{Z}^+$, $0 < t \leq |\mathcal{A}^{(i)}|$,
such that $|\triangle_N L(t, A^{(h)})| \geq t$. The antichain $A$ is non-reducible if for each $t \in \mathbb{Z}^+$, $0 < t \leq |A^{(h)}|$, $|\triangle_N L(t, A^{(h)})| < t$. That is, $A$ is reducible if, for some $t$, $0 < t \leq |A^{(h)}|$, the last $t$-sets of $A^{(h)}$ can be replaced by $t$ $(h - 1)$-sets preceding the sets in $A^{(h-1)}$ in squashed order while preserving the antichain property. If such a replacement is not possible, then $A$ is non-reducible.

Further, $A$ is said to be full if either $A^{(i)}$ consists of the last $i$-subsets of $[n]$ in squashed order or $A = F_{n,i}(\mathcal{A})$ and $|\mathcal{I}^{(i)}(A)| = \binom{n}{i+1}$.

**Example 1.10.** $\{1234, 125, 35\}$ is a squashed antichain and $\{5, 34, 124\}$ is an antilexicographic antichain on $[5]$.

$\{12, 13, 23, 4, 5, 6\}$ is reducible as $\{1, 2, 3, 4, 5, 6\}$ is an antichain. $\{12, 13, 23, 14, 24, 5, 6\}$ is non-reducible.

$\{123, 124, 34, 15, 25, 35, 45\}$ and $\{123, 124, 134, 234, 125, 135, 145\}$ are both full antichains on $[5]$.

Let $A$ and $B$ be antichains on $[n]$ with $|A| = |B|$ and $V(A) = V(B)$. We say that $A$ and $B$ are profile-equivalent, denoted by $A \cong B$, if $A$ and $B$ have the same profile.

Let $P$ be some property of antichains and let $A$ be an antichain with property $P$. If $A \cong B$ for each antichain $B$ with property $P$, then we say that $A$ is a profile-unique antichain with property $P$.

**Example 1.11.** The antichains $A = \{123, 124, 134, 15\}$ and $B = \{134, 125, 135, 45\}$ on $[5]$ both have profile $(0, 0, 1, 3, 0, 0)$. Thus $A \cong B$. Note that $A \neq B$. Let $P$ be the property that the antichains are flat and have size 4 and volume 11. Then $A$ and $B$ are profile-unique antichains with property $P$.

Let $\Lambda_n$ denote the collection of all antichains on $[n]$. That is, $\Lambda_n = \{A : A$ is an antichain on $[n]\}$. For antichains in $\Lambda_n$ we define $V_{\max}(\Lambda_n) = \max_{A \in \Lambda_n} V(A)$, and $V_{\max}(\Lambda_n) = \max_{A \in [n]} \{V : V = V(A) \text{ where } A = [n]^k, \ V < V_{\max}(\Lambda_n)\}$.

Let $\Lambda_{n,s}$ denote the collection of all antichains on $[n]$ of size $s$. That is, $\Lambda_{n,s} = \{A : A \in \Lambda_n, |A| = s\}$. For antichains in $\Lambda_{n,s}$ we define $V_{\min}(\Lambda_{n,s}) = \min_{A \in \Lambda_{n,s}} V(A)$, $V_{\max}(\Lambda_{n,s}) = \max_{A \in \Lambda_{n,s}} V(A)$, and $I_{\min}(\Lambda_{n,s}) = \min_{A \in \Lambda_{n,s}} |z(A)|$. We say that $A \in \Lambda_n$
\( \Lambda_{n,s} \) achieves minimum (or maximum) volume in \( \Lambda_{n,s} \) if \( V(A) = V_{\min}(\Lambda_{n,s}) \) or \( V(A) = V_{\max}(\Lambda_{n,s}) \). Similarly we say that \( A \in \Lambda_{n,s} \) achieves minimum size ideal in \( \Lambda_{n,s} \) if \( |I_A| = I_{\min}(\Lambda_{n,s}) \).

### 1.2.4 A Miscellany

**Binomial Representations**

Let \( p, k \in \mathbb{Z}^+ \). The \( k \)-binomial representation of \( p \) is defined by \( p = \binom{a_t}{k} + \binom{a_{t-1}}{k-1} + \ldots + \binom{a_1}{1} \) where \( a_t > a_{t-1} > \ldots > a_1 \geq 1 \). The \( k \)-binomial representation of a number is unique as seen in Theorem 2.12.

The real \( k \)-binomial representation of \( p \) is defined by \( p = \binom{x}{k} \) where \( x \in \mathbb{R}^+ \), \( x \geq k \), and \( \binom{x}{k} \) is defined to be \( \binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!} \).

**Example 1.12.** The real binomial representations given in this example are correct to three decimal places.

Let \( k = 4 \) and \( p = 7 \). Then \( p = \binom{5}{4} + \binom{3}{3} + \binom{2}{2} \) is the 4-binomial representation of 7 and \( p = \binom{5.275}{4} \) is the real 4-binomial representation of 7.

Let \( p = 30 \). Then \( p = \binom{7}{4} + \binom{5}{3} + \binom{2}{2} + \binom{1}{1} \) is the 4-binomial representation of 30 and \( p = \binom{6.801}{4} \) is the real 4-binomial representation of 30.

**Completely Separating Systems**

The collection \( C \) of subsets of \([n]\) is a completely separating system if for each ordered pair \((i,j) \in [n] \times [n] \) there exists \( C \in C \) such that \( i \in C \) and \( j \notin C \). A completely separating system on \([n]\) is denoted by \((n)CSS\). A minimal \((n)CSS\) is the smallest size completely separating system on \([n]\). The size of a minimal \((n)CSS\) is denoted by \( R(n) \). If the sets in a \((n)CSS\) are all \( k \)-sets, then the completely separating system is denoted by \((n,k)CSS\). The size of a minimal \((n,k)CSS\) is denoted by \( R(n,k) \).

Let \( C = \{C_1, \ldots, C_m\} \) be a completely separating system on \([n]\). The dual \( A \) of \( C \)
is the collection $\mathcal{A} = \{X_1, \ldots, X_n\}$ of subsets of $[m]$ such that $X_i = \{j : i \in C_j, 1 \leq j \leq m\}$ for all $i \in [n]$.

**Example 1.13.** $\mathcal{C} = \{124, 135, 236, 456\}$ is a minimal $(6,3)$CSS. The dual of $\mathcal{C}$ is $\{12, 13, 23, 14, 24, 34\}$.

**Multisets**

A generalisation of a set is a multiset. Multisets are not the object of this thesis. However they will be mentioned while discussing some results for sets. A multiset on $[n]$ consists of collections of elements of $[n]$ where every element may occur more than once in the collections. Let $k_1, k_2, \ldots, k_n \in \mathbb{N}$. The multiset $S(k_1, k_2, \ldots, k_n)$ is defined to consist of all the collections of elements of $[n]$ which contain at most $k_i$ occurrences of the element $n - i + 1$, $i \in [n]$. An element $m$ of $S(k_1, k_2, \ldots, k_n)$ is denoted by the vector $m = (m_1, m_2, \ldots, m_n)$, $m_i \leq k_i$, where $m_i$ denotes the number of occurrences of the element $n - i + 1$ in $m$. The rank of $m$ is $|m| = \sum_{i=1}^{n} m_i$. A vector of rank $k$ is called a $k$-vector.

The lexicographic order on $S(k_1, k_2, \ldots, k_n)$, denoted by $\leq_L$, is defined by: Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two elements of $S(k_1, k_2, \ldots, k_n)$. Then $a \leq_L b$ if $a_1 < b_1$, or if $a_1 = b_1$, $\ldots$, $a_{i-1} = b_{i-1}$, $a_i < b_i$ for some $i$, $1 < i \leq n$, or if $a = b$.

**Example 1.14.** Consider the multiset $S(2, 3, 4)$. The vectors of rank 3 in lexicographic order are (the vectors are written with the brackets left out): 003, 012, 021, 030, 102, 111, 120, 201, 210.

If $\mathcal{M}$ is a collection of $k$-vectors of $S(k_1, k_2, \ldots, k_n)$, then the shadow of $\mathcal{M}$ is $\Delta \mathcal{M} = \{m = (m_1, m_2, \ldots, m_n) : |m| = k - 1; (m_1, \ldots, m_{i-1}, m_i+1, \ldots, m_n) \in \mathcal{M}$ for some $i, 1 \leq i \leq n\}$.

**Convex Functions**

The function $f : D \to \mathbb{R}$ is said to be convex if $f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b)$ for all $a, b \in D$ and all $\lambda \in \mathbb{R}^+ \cup \{0\}$ such that $\lambda a + (1-\lambda)b \in D$ and $0 \leq \lambda \leq 1$. 


Chapter 2

Background Material & Survey of Results about Shadows and Shades


2.1 Introduction

This chapter provides the background material required in the later chapters. It also surveys what is known about shadows, shades, new-shadows and new-shades of collections of sets. Some results are fundamental results while some others are simple results which are included here to ease the flow of discourse in the later chapters.

Most results in this chapter are found in the existing literature. For a more detailed study of Sperner theory and antichains see [1, 2, 3, 12, 13]. Most theorems and lemmas are referenced and have the name(s) of their author(s) attached to them. Results without references or author names are well-known results which may not appear in the literature in the form in which they are stated in the present chapter. Proofs will be given for these results, as well as for new results. In general, proofs are given whenever they are thought to help understanding.

New results are marked by the symbol $\Diamond$. In this and the later chapters, most results will be referred to by their numbering. However, some results will be referred to by their name. Table C.1 in Appendix C gives the correspondence between names and numbering for these results.

The purpose of Sections 2.2 to 2.10 is to survey various aspects of the squashed order and of shadows and shades of collections of sets. Section 2.2 states simple observations about the squashed order and shadows, shades, new-shadows, and new-shades of collections of sets. Section 2.3 considers the relationship between the $k$-binomial representation of a number and the squashed order.

Section 2.4 establishes a relationship between the shadow of a collection of sets $B$ and the shade of the complement of $B$. In Section 2.5 it is shown that shadows, shades, new-shadows, and new-shades preserve the squashed order. Section 2.6 considers collections of sets for which there exists a correspondence between the collections.

In Sections 2.7 and 2.8 lower and upper bounds for the sizes of shadows, shades, new-shadows, and new-shades are given. Section 2.9 investigates the sizes of the new-shadow and new-shade of a set.
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Let \( B \) be a collection of consecutive sets of the same size in squashed order. In Section 2.10 relationships between \( \Delta_N(\nabla_N B) \) and \( B \) and between \( \nabla_N(\Delta_N B) \) and \( B \) are made.

Section 2.11 concerns antichains. Some important known results about antichains are stated there. The chapter concludes with Section 2.12 which presents basic results about completely separating systems and convex functions.

### 2.2 Simple Facts about the Squashed Order and Shadows, Shades, New-Shadows, and New-Shades

In this section are stated simple observations arising from the definitions of the squashed order and shadows, shades, new-shadows, and new-shades. Let \( A \) and \( B \) be two sets such that \( A \leq_S B \) and recall that the antilexicographic order is the reverse of the squashed order. Since \( A + B = A' + B' \), \( A \leq_S B \) implies that \( B' \leq_S A' \) and \( A' \leq_A B' \). This fact and some of its consequences are summarised in the next four observations.

**Observation 2.1.** \( A \leq_S B \) if and only if \( B' \leq_S A' \). That is, \( A \leq_S B \) if and only if \( A' \leq_A B' \).

**Observation 2.2.** \( B \) is a collection of sets in squashed order if and only if \( B' \) is a collection of sets in antilexicographic order.

**Observation 2.3.** \( (F_{n,k}(p))' = L_{n,n-k}(p) \).

**Observation 2.4.** \( B \) is the \( p \)-th \( k \)-subset of \([n]\) in squashed order if and only if \( B' \) is the \( p \)-th \((n-k)\)-subset of \([n]\) in antilexicographic order.

**Example 2.5.** The first column below lists the 3-subsets of [5] in squashed order; the second column lists the complements of the 3-sets giving the list of the 2-subsets in antilexicographic order.
Note that the squashed order is independent of the universal set. This implies that $F_{n,k}(p) = F_{n',k}(p)$ for any $n'$ such that $p \leq \binom{n'}{k}$. We state this as an observation.

**Observation 2.6.** Let $p \leq \binom{n-1}{k}$. Then $F_{n,k}(p) = F_{n-1,k}(p)$.

Given the definition of $F_{n,k}$, $C_{n,k}$ and $L_{n,k}$, it is easy to see that

**Observation 2.7.**

$$F_{n,k}\left(\binom{n}{k}\right) = C_{n,k}\left(\binom{n}{k}\right) = L_{n,k}\left(\binom{n}{k}\right) = \lfloor n \rfloor^k.$$  

It follows that

**Observation 2.8.**

$$|\nabla F_{n,k}\left(\binom{n}{k}\right)| = \binom{n}{k-1},$$  

$$|\nabla L_{n,k}\left(\binom{n}{k}\right)| = \binom{n}{k+1}.$$  

The next observations follow from the definitions of the new-shadow and the new-shade.
Observation 2.9.

\[ \Delta_N F_{n,k}(p) = \Delta F_{n,k}(p), \]
\[ \nabla_N L_{n,k}(p) = \nabla L_{n,k}(p). \]

Observation 2.10. If \( A \) and \( B \) are two collections of \( k \)-sets such that \( A \cap B = \emptyset \), then

\[ |\Delta_N (A \cup B)| = |\Delta_N A| + |\Delta_N B|, \]
\[ |\nabla_N (A \cup B)| = |\nabla_N A| + |\nabla_N B|. \]

Observation 2.11.

\[ |\Delta F_{n,k}(p_1 + p_2)| = |\Delta F_{n,k}(p_1)| + |\Delta_N N_{n,k}^{p_1}(p_2)|, \]
\[ |\Delta L_{n,k}(p_1 + p_2)| = |\Delta L_{n,k}(p_1)| + |\Delta_N N_{n,k}^{p_1}(p_2)|, \]
\[ |\nabla F_{n,k}(p_1 + p_2)| = |\nabla F_{n,k}(p_1)| + |\nabla_N N_{n,k}^{p_1}(p_2)|, \]
\[ |\nabla L_{n,k}(p_1 + p_2)| = |\nabla L_{n,k}(p_1)| + |\nabla_N N_{n,k}^{p_1}(p_2)|. \]

2.3 The \( k \)-Binomial Representation and the Squashed Order

The \( k \)-binomial representation of an integer \( p \) is related to the squashed ordering of \( p \) \( k \)-sets as will become apparent in Lemma 2.15 below. The \( k \)-binomial representation of a number is unique.

Theorem 2.12 ([1, p. 115]). Let \( p, k \in \mathbb{Z}^+ \). Then \( p \) has a unique \( k \)-binomial representation.

The next two lemmas are simple results which determine the \( k \)-binomial representation of \( p + 1 \) by using the \( k \)-binomial representation of \( p \).

Lemma 2.13. Let \( p \in \mathbb{Z}^+ \). Assume that the \( k \)-binomial representation of \( p \) is \( \sum_{i=1}^{k} \binom{a_i}{i} \). Let \( j \) denote the smallest \( i \) for which \( a_{i+1} \neq a_i + 1 \). Then \( \binom{a_j}{j} + \sum_{i=j+1}^{k} \binom{a_i}{i} \) is the \( k \)-binomial representation of \( p + 1 \).
Proof. Let $p$ and $j$ be as in the statement of the lemma. The $k$-binomial representation of $p$ is $p = \left( \binom{a_k}{k} \right) + \ldots + \left( \binom{a_{i+1}}{j+1} \right) + \left( \binom{a_{j}}{j} \right) + \ldots + \left( \binom{a_{1}}{1} \right) + 1$. Then
\[
p + 1 = \left( \binom{a_k}{k} \right) + \ldots + \left( \binom{a_{j+1}}{j+1} \right) + \left( \binom{a_{j}}{j} \right) + \ldots + \left( \binom{a_{1}}{1} \right) + 1
= \left( \binom{a_k}{k} \right) + \ldots + \left( \binom{a_{j+1}}{j+1} \right) + \left( \binom{a_{j}}{j} \right) + \ldots + \left( \binom{a_{1}+1}{1} \right)
= \left( \binom{a_k}{k} \right) + \ldots + \left( \binom{a_{j+1}}{j+1} \right) + \left( \binom{a_{j}+1}{j} \right)
\]
By assumption $a_{j+1} > a_j$ and $a_{j+1} \neq a_j + 1$. Thus $a_{j+1} > a_j + 1$. Recall that $a_{i+1} > a_i$ for $i = j + 1, \ldots, k - 1$. It follows that $\left( \binom{a_k}{k} \right) + \ldots + \left( \binom{a_{j+1}}{j+1} \right) + \left( \binom{a_{j}+1}{j} \right)$ is the $k$-binomial representation of $p + 1$ by Theorem 2.12.

Lemma 2.14. Let $p \in \mathbb{Z}^+$. Assume that the $k$-binomial representation of $p$ is $\sum_{i=1}^{k} \binom{a_i}{i}$ with $t > 1$. Then $\sum_{i=1}^{k} \binom{a_i}{i} + \binom{a_{i+1}}{i+1}$ is the $k$-binomial representation of $p + 1$.

Proof. Let $p$ and $j$ be as in the statement of the lemma. The $k$-binomial representation of $p$ is $p = \left( \binom{a_k}{k} \right) + \ldots + \left( \binom{a_{i}}{i} \right)$. Then
\[
p + 1 = \left( \binom{a_k}{k} \right) + \ldots + \left( \binom{a_{i}}{i} \right) + 1
= \left( \binom{a_k}{k} \right) + \ldots + \left( \binom{a_{i}}{i} \right) + \binom{t - 1}{t - 1}
\]
By assumption $a_{t} \geq t > 1$. Thus $a_{t} > t - 1 \geq 1$ and it follows that $\left( \binom{a_k}{k} \right) + \ldots + \left( \binom{a_{i}}{i} \right) + \binom{a_{t+1} - 1}{t-1}$ is the $k$-binomial representation of $p + 1$ by Theorem 2.12.

There is a close relationship between the $k$-binomial representation of $p$ and the squashed order. This is shown in the next lemma.

Lemma 2.15. Let $\sum_{i=1}^{k} \binom{a_i}{i}$ be the $k$-binomial representation of $p$. Then
\[
F_{n,k}(p) = \left[ a_k \right]^{k} \cup \left( \left[ a_{k-1} \right]^{[i-1]} \sqcup \left( \left[ a_{k-1} \right]^{[i-2]} \sqcup \left( \left[ a_{k-1} \right]^{[i-3]} \sqcup \ldots \sqcup \left[ a_{i-1} \right] \sqcup \left[ a_{i} \right] \sqcup \left[ a_{i+1} \right] \sqcup \ldots \sqcup \left[ a_{k} \right] \right) \right) \right]
\]
Proof. This proof is adapted from [1, p. 117]. As few elements as possible are used when listing sets in squashed order. Let \( \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t} \) be the \( k \)-binomial representation of \( p \). We describe \( F_{n,k}(p) \). Recall that \( a_k > a_{k-1} > \ldots > a_t \geq t \geq 1 \).

\( F_{n,k}(p) \) consists of all the \( k \)-subsets of \([a_k]\), followed by all the \((k-1)\)-subsets of \([a_{k-1}]\) to which the set \([a_k + 1]\) has been adjoined, followed by all the \((k-2)\)-subsets of \([a_{k-2}]\) to which the set \([a_{k-1} + 1, a_k + 1]\) has been adjoined, and so on. In other words, \( F_{n,k}(p) \) is the union of the non-intersecting collections \([a_k]^k\), \([a_{k-1}]^{(k-1)} \cup \{a_k + 1\}\), \([a_{k-2}]^{(k-2)} \cup \{a_{k-1} + 1, a_k + 1\}\), \ldots, \([a_t]^t \cup \{a_{t+1} + 1, \ldots, a_k + 1\}\).

Example 2.16. Let \( n = 5, k = 3, \) and \( p = 7 \). Then \( p = \binom{3}{3} + \binom{2}{2} \) and \( F_{5,3}(7) = \{123, 124, 134, 234, 125, 135, 235\} = [4]^3 \cup ([3]^2 \cup \{5\}) \).

Let \( p = 9 \). Then \( p = \binom{3}{3} + \binom{2}{2} + \binom{1}{1} \) and \( F_{5,3}(9) = \{123, 124, 134, 234, 125, 135, 235, 145, 245\} = [4]^3 \cup ([3]^2 \cup \{5\}) \cup ([2]^1 \cup \{4, 5\}) \).

2.4 Relationships between \( \Delta B \) and \( \nabla B' \) and between \( \Delta_N B \) and \( \nabla_N B' \)

If \( B \) is a collection of sets in squashed order, then \( B' \) is a collection of sets in antilexicographic order by Observation 2.2. In this section the relationship between \( B \) and \( B' \) is further explored in terms of \( \Delta B \) and \( \nabla B' \). If \( A \) and \( B \) are two sets such that \( A \subseteq B \) then \( B' \subseteq A' \).

Lemma 2.17. Let \( B \) be a subset of \([n]\) and let \( B \) be a collection of \( k \)-subsets of \([n]\). Then

(i) \( (\Delta B)' = \nabla B' \), and

(ii) \( (\Delta B)' = \nabla B' \).

Proof. Let \( A \) and \( B \) be a \((k - 1)\) and a \( k \)-subset of \([n]\) respectively. \( A \in \Delta B \) implies that \( A' \supseteq B' \), where \( A' \) and \( B' \) are a \((n-k+1)\) and a \((n-k)\)-subset of \([n]\) respectively. Thus \( A' \in \nabla B' \). Conversely, if \( A' \in \nabla B' \) then \( A \in \Delta B \). This proves (i) from which (ii) follows as \( (\Delta B)' = \bigcup_{B \in \Delta} (\Delta B)' = \bigcup_{B \in \Delta} \nabla B' = \nabla B' \).
Lemma 2.18.

\[ |\Delta F_{n,k}(p)| = |\nabla L_{n,n-k}(p)|. \]

Proof. By Lemma 2.17 and Observation 2.3, \[ |\Delta F_{n,k}(p)| = |(\Delta F_{n,k}(p))'| = |\nabla L_{n,n-k}(p)|. \]

Example 2.19. \( \Delta F_{5,3}(3) = \{123, 124, 134\} = \{12, 13, 23, 14, 24, 34\} \) and \( \nabla L_{5,2}(3) = \nabla \{25, 35, 45\} = \{125, 135, 235, 145, 245, 345\} = (\Delta \{123, 124, 134\})' \).

A similar result to that of Lemma 2.17 holds for new-shadows and new-shades.

Lemma 2.20. Let \( B \) be a subset of \([n]\) and let \( \mathcal{B} \) be a collection of \( k \)-subsets of \([n]\). Then

(i) \( (\nabla N B)' = \nabla N B' \), and

(ii) \( (\Delta N B)' = \nabla N B' \).

Proof. Let \( B \) be a \( k \)-subset of \([n]\) such that \( B \) is the \( p \)th set in squashed order. Then

\[
(\Delta N B)' = \left( (\Delta F_{n,k}(p)) \setminus (\Delta F_{n,k}(p-1)) \right)'
= \left( (\nabla L_{n,n-k}(p))' \setminus (\nabla L_{n,n-k}(p-1))' \right)'
\]

by Lemma 2.17 and Observation 2.3. It follows that

\[
(\Delta N B)' = (\nabla L_{n,n-k}(p)) \setminus (\nabla L_{n,n-k}(p-1))
= \nabla N B'
\]

by Observation 2.4. This proves (i). Then (ii) follows by Observation 2.10.

The next lemma is a consequence of Lemma 2.20.

Lemma 2.21. Let \( B \) be a subset of \([n]\) and let \( \mathcal{B} \) be a collection of \( k \)-subsets of \([n]\). Then

(i) \( |\Delta N B| = |\nabla N B'| \), and

(ii) \( |\Delta N B| = |\nabla N B'| \).
Proof. Take cardinalities in Lemma 2.20 and note that $|\triangle_N B| = |(\triangle_N B)'|$ and $|\triangle_N B| = |(\triangle_N B)'|$. \hfill \qed

Lemma 2.22. $|\triangle_N L_{n,k}(p)| = |\nabla_N F_{n,n-k}(p)|$.

Proof. The result follows from Observation 2.3 and Lemma 2.21.(ii). \hfill \qed

2.5 Shadows, Shades, New-Shadows and New-Shades Preserve the Squashed Order

The shadow of an initial segment of $k$-sets in squashed order is an initial segment of $(k-1)$-sets in squashed order. Similarly, the shade of a terminal segment of $k$-subsets of $[n]$ in squashed order is a terminal segment of $(k+1)$-subsets of $[n]$ in squashed order.

Lemma 2.23 ([13, p. 103]).

(i) $\triangle F_{n,k}(p) = F_{n,k-1}(\triangle F_{n,k}(p))$, and

(ii) $\nabla L_{n,k}(p) = L_{n,k+1}(\nabla L_{n,k}(p))$.

Proof. Only (i) is stated and proved in [13, p. 103]. (ii) follows from (i) by replacing $k$ by $n-k$ and applying Lemmas 2.17 and 2.18. \hfill \qed

Example 2.24. $F_{5,3}(5) = \{123, 124, 134, 234, 125\}$ and $\triangle F_{5,3}(5) = \{12, 13, 23, 14, 24, 34, 15, 25\} = F_{5,2}(\triangle F_{5,3}(5))$. \hfill \qed

New-shadows and new-shades preserve the squashed order. This is shown below.

Lemma 2.25. Let $A$ and $B$ be $k$-sets such that $A \leq_S B$, and $\triangle_N A$, $\triangle_N B$, $\nabla_N A$, $\nabla_N B \neq \emptyset$. Then

(i) $\triangle_N A \leq_S \triangle_N B$,

(ii) $\nabla_N A \leq_S \nabla_N B$. 

Proof. This follows from the properties of shadows and shades. \hfill \qed
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Proof. Let $A$ and $B$ be as in the statement of the lemma. We prove (i); (ii) can be proved by using a similar argument. Let $A$ be the $p$th $k$-set in squashed order. By Lemma 2.23, $\triangledown F_{n,k}(p) = F_{n,k-1}(\triangledown F_{n,k}(p))$, so that $\Delta_N B > S F_{n,k-1}(\triangledown F_{n,k}(p))$ by the definition of the new-shadow. That is, $\Delta_N B > S \Delta_N A$ as $\Delta_N A \subseteq \Delta F_{n,k}(p)$. \hfill \Box

Lemmas 2.26 and 2.27 are direct consequences of Lemma 2.25.

Lemma 2.26. If $B$ is a collection of consecutive $k$-sets in squashed order, then $\Delta_N B$ and $\triangledown_N B$ are collections of consecutive sets in squashed order.

Lemma 2.27. 

(i) $\Delta_N L_{n,k}(p) = L_{n,k-1}(\triangledown_N L_{n,k}(p))$, and

(ii) $\triangledown_N F_{n,k}(p) = F_{n,k+1}(\triangledown_N F_{n,k}(p))$.

2.6 Some Correspondence Results

The three lemmas in this section are obtained by establishing a correspondence between a collection of $p$ subsets of $[n]$ in squashed order and a collection of $p$ subsets of $[n-i]$ in squashed order for $0 < i < n$. This is possible when $p$ is small. For convenience, in each lemma the meaning of correspondence between collections of sets is restated in terms of the new-shadows and the new-shades of the collections of sets. The first lemma follows from the observation that the squashed order of sets is independent of the universal set.

Lemma 2.28. Let $0 \leq i \leq n - k$ and $p \leq \binom{n-i}{k}$. Then the collections $F_{n,k}(p)$ and $F_{n-i,k}(p)$ correspond. That is,

$|\Delta_N F_{n,k}(p)| = |\Delta_N F_{n-i,k}(p)|$, and

$|\triangledown_N F_{n,k}(p)| = |\triangledown_N F_{n-i,k}(p)|$.

Proof. This follows from Observation 2.6. \hfill \Box

Example 2.29. For $n = 5$, $k = 3$ and $p = 3$, $F_{5,3}(3) = \{123, 124, 134\} = F_{4,3}(3)$ so that $\Delta_N F_{5,3}(3) = \Delta_N F_{4,3}(3)$.
The following lemma is based upon an idea of Clements [8].

**Lemma 2.30.** Let \( B \) be a collection of consecutive \((k + m)\)-subsets of \([n]\) in squashed order. Assume that \( B = C \uplus L \) where \( L \) is the set \( \{l_1, l_2, \ldots, l_m\} \), \( l_1 < l_2 < \ldots < l_m \). Assume also that \( C \cup \{l_1\} \neq [k + 1] \) for each \( C \in C \). Then \( B \) and \( C \) correspond. That is,

\[
|\triangle_{N}B| = |\triangle_{N}C|, \quad \text{and}
\]
\[
|\nabla_{N}B| = |\nabla_{N}C|.
\]

**Proof.** First note that \( |B| = |C| \). We begin by showing that \( |\triangle_{N}B| = |\triangle_{N}C| \).

**Claim 1.** If \( D \in \triangle_{N}B \) then \((D \setminus L) \in \triangle_{N}C\).

**Proof.** Let \( D \in \triangle_{N}B \). Let \( B \in B \) be the \((k + m)\)-set such that \( D \in \triangle_{N}B \). We show that \( L \subseteq D \). Assume that \( L \not\subseteq D \) and let \( l \) be the element in \( L \setminus D \) so that \( B = D \cup \{l\} \).

Recall that by the definition of the operator \( \uplus \) any element in \( L \) is larger than any element in the sets in \( C \). Thus if \( c \) denotes the largest element in the sets in \( C \) it follows that \( c < l_1 \). Since \( C \cup \{l_1\} \neq [k + 1] \) for each \( C \in C \), \( l_1 < l_2 < \ldots < l_m \) by definition, and \( c < l_1 \), there exists an element \( d \notin B \) such that \( d < l_1 \). It follows that \( d < l \). Let \( B_1 \) be the \((k + m)\)-set \( D \cup \{d\} \). Note that \( B_1 + B = \{d, l\} \) so that \( B_1 <_{S} B \) since \( d < l \). As \( D \subseteq B_1 \) we have that \( D \notin \triangle_{N}B \), contradicting the assumption about \( D \). This shows that \( L \subseteq D \).

As \( L \subseteq D \), \( |D \setminus L| = k - 1 \). We now show that \((D \setminus L) \in \triangle_{N}C\). Assume that \((D \setminus L) \notin \triangle_{N}C \). If \((D \setminus L) \notin \triangle_{N}C \) then there exists a \( k \)-set \( A \) such that \((D \setminus L) \subset A \) and \( A <_{S} C \). The observation that \( A <_{S} C \) follows from the fact that \((D \setminus L) \subset (B \setminus L) \in C \) since \( D \subset B \in B \). Also \( A \cap L = \emptyset \) as \( A <_{S} C \) and \( c < l_1 \) so that \( A \cup L \) is a \((k + m)\)-set. It follows that \( D \subset (A \cup L) \) where \((A \cup L) <_{S} B \). This contradicts the assumption that \( D \in \triangle_{N}B \) and proves Claim 1.

Using a similar proof by contradiction to the one in the last paragraph one can show that if \( D \in \triangle_{N}C \) then \((D \uplus L) \in \triangle_{N}B \). The details are left to the reader. Thus \( |\triangle_{N}B| = |\triangle_{N}C| \). Next we show that \( |\nabla_{N}B| = |\nabla_{N}C| \).
Claim 2. If $D \in \nabla_N B$ then $(D \setminus L) \in \nabla_N C$.

Proof. Assume that $D \in \nabla_N B$ and $(D \setminus L) \not\in \nabla_N C$. If $(D \setminus L) \not\in \nabla_N C$ then there exists a $k$-set $A$ such that $A \subset (D \setminus L)$ and $A >_S C$. The observation that $A >_S C$ follows from the fact that $(D \setminus L) \supset C$ for some $C \in \mathcal{C}$ since $D \in \nabla_N B$. This implies that $(A \cup L) \subset D$ where $(A \cup L) >_S B$. This contradicts the assumption that $D \in \nabla_N B$ and proves Claim 2.

Using a similar proof by contradiction one can show that if $D \in \nabla_N C$ then $(D \uplus L) \in \nabla_N B$. Thus $|\nabla_N B| = |\nabla_N C|$. This concludes the proof of Lemma 2.30. \hfill \qed

See Example 1.7 for an illustration of Lemma 2.30. The next lemma now follows easily from Lemma 2.30.

Lemma 2.31. Let $0 \leq i \leq k$ and $p \leq \binom{n-1}{k-i}$. Then the collections $L_{n,k}(p)$ and $L_{n-i,k-i}(p)$ correspond. That is,

\[|\triangle_N L_{n,k}(p)| = |\triangle_N L_{n-i,k-i}(p)|, \text{ and} \]
\[|\nabla_N L_{n,k}(p)| = |\nabla_N L_{n-i,k-i}(p)|.\]

Proof. Let $0 \leq i \leq k$ and $p \leq \binom{n-1}{k-i}$. As $p \leq \binom{n-1}{k-i}$ any set $A \in L_{n,k}(p)$ contains the $i$-set $\{n-i+1, n-i+2, \ldots, n\}$. Thus $L_{n,k}(p) = L_{n-i,k-i}(p \uplus \{n-i+1, n-i+2, \ldots, n\}$ and $L_{n,k}(p)$ and $L_{n-i,k-i}(p)$ correspond by Lemma 2.30. \hfill \qed

Example 2.32. Let $n = 5$, $k = 3$ and $p = 4$. Note that $4 \leq \binom{4}{3}$. Then $L_{5,3}(4) = \{235, 145, 245, 345\}$ corresponds to $\{23, 14, 24, 34\} = L_{4,2}(4)$. Here $\triangle_N L_{5,3}(4) = \{45\}$, $\triangle_N L_{4,2}(4) = \{4\}$, $\nabla_N L_{5,3}(4) = \{1235, 1245, 1345, 2345\}$, and $\nabla_N L_{4,2}(4) = \{123, 124, 134, 234\}$. Thus $|\triangle_N L_{5,3}(4)| = |\triangle_N L_{4,2}(4)|$ and $|\nabla_N L_{5,3}(4)| = |\nabla_N L_{4,2}(4)|$. \hfill \qed

2.7 Bounds for Shadows and Shades

In this section lower and upper bounds for the sizes of the shadow and the shade of a collection of sets are given. Sperner's lemma below gives a lower bound for the sizes
of the shadow and the shade of a collection \( \mathcal{B} \) which is expressed in terms of the size of \( \mathcal{B} \).

**Lemma 2.33 (Sperner’s lemma, Sperner [30]).** Let \( \mathcal{B} \) be a collection of \( k \)-subsets of \([n]\). Then

\[
|\triangle \mathcal{B}| \geq \frac{k}{n-k+1} |\mathcal{B}| \quad \text{if } k > 0
\]

and

\[
|\nabla \mathcal{B}| \geq \frac{n-k}{k+1} |\mathcal{B}| \quad \text{if } k < n.
\]

It directly follows that

**Lemma 2.34.** If \( k \leq \frac{n-1}{2}, \) \( |\nabla \mathcal{B}| \geq |\mathcal{B}|. \) If \( k \geq \frac{n+1}{2}, \) \( |\triangle \mathcal{B}| \geq |\mathcal{B}|. \)

Equality in Sperner’s lemma holds in one case only.

**Lemma 2.35 ([3, p. 12]).** Equality holds in Lemma 2.33 if and only if \( \mathcal{B} \) consists of all the \( \binom{n}{k} \) \( k \)-sets.

The next theorem by Kruskal and Katona shows that an initial segment of \( p \) \( k \)-sets in squashed order minimises the size of the shadow over all collections of \( p \) \( k \)-sets.

**Theorem 2.36 (Kruskal [18], Katona [14]).** Let \( \mathcal{B} \) be a collection of \( k \)-subsets of \([n]\). Then

\[
|\triangle \mathcal{B}| \geq |\triangle F_{n,k}(|\mathcal{B}|)|.
\]

**Corollary 2.37.** If \( \mathcal{B} \) is a collection of \( k \)-subsets of \([n]\) then

\[
|\nabla \mathcal{B}| \geq |\nabla L_{n,k}(|\mathcal{B}|)|.
\]

**Proof.** Let \( \mathcal{B} \) be a collection of \( k \)-subsets of \([n]\), then \( \mathcal{B}' \) is a collection of \((n-k)\) subsets of \([n]\). By Theorem 2.36, \( |\triangle \mathcal{B}'| \geq |\triangle F_{n,n-k}(|\mathcal{B}'|)|. \) By Lemmas 2.17 and 2.18 it follows that \( |\nabla \mathcal{B}| \geq |\nabla L_{n,k}(|\mathcal{B}|)| \) as required. \(\square\)
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Corollary 2.37 shows that a terminal segment of \( p \) \( k \)-subsets of \([n] \) in squashed order minimises the size of the shade over all collections of \( p \) \( k \)-sets. That is, an initial segment of \( p \) \( k \)-subsets of \([n] \) in antilexicographic order is a collection whose shade has minimum size. Note that although Theorem 2.36 determines one collection of \( p \) \( k \)-sets whose shadow has minimum size, it still remains to find all collections of \( p \) \( k \)-sets whose shadow has minimum size. This is stated as an open problem.

**Open Problem 2.38 ([13, p. 101])**. Determine all collections of \( p \) \( k \)-subsets of \([n] \) which minimise the size of the shadow of a collection of \( p \) \( k \)-subsets of \([n] \).

Theorem 2.36 can be equivalently restated as

**Theorem 2.39.** Let \( \mathcal{B} \) be a collection of \( p \) \( k \)-subsets of \([n] \). Assume that the \( k \)-binomial representation of \( p \) is

\[
p = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}.
\]

Then

\[
|\Delta \mathcal{B}| \geq \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_t}{t-1}.
\]

Equality holds when \( \mathcal{B} \) is an initial segment of \( k \)-sets in squashed order.

Theorem 2.39 is the original formulation of Theorem 2.36. That Theorem 2.39 implies Theorem 2.36 is easy to see. We show that Theorem 2.36 implies Theorem 2.39.

**Proof of Theorem 2.39.** The proof is an adaptation of [1, p. 119]. Let \( p = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t} \) be the \( k \)-binomial representation of \( p \). By Lemma 2.15, \( F_{n,k}(p) \) is the union of the non-intersecting collections \( [a_k]^k, [a_{k-1}]^{k-1} \cup \{a_{k+1}\}, [a_{k-2}]^{k-2} \cup \{a_{k+1}, a_{k+1}\}, \ldots, [a_t]^t \cup \{a_{t+1}, \ldots, a_{k+1}\} \). Denote the collection \( [a_k]^k \) by \( \mathcal{B}_k \), and the collections \( [a_i]^i \cup \{a_{i+1}, \ldots, a_{k+1}\}, i = k-1, k-2, \ldots, t \), by \( \mathcal{B}_i \). Note that \( |\Delta F_{n,k}(p)| = \sum_{i=1}^k |\Delta \mathcal{B}_i| \) by Observations 2.11 and 2.9.

By Lemma 2.30, \( |\Delta \mathcal{B}_i| = |\Delta_N([a_i]^i)| \) for \( i = t, \ldots, k-1 \). Since \( [a_i]^i = F_{n,i}(\binom{a_i}{i}) \) by Observation 2.7, it follows that \( |\Delta F_{n,k}(p)| = \sum_{i=1}^k |\Delta_N([a_i]^i)| = \sum_{i=1}^k \binom{a_i}{i-1} \) by Observations 2.9 and 2.8. Theorem 2.39 now follows from Theorem 2.36. \(\square\)
Example 2.40. As in Example 2.16 let \( n = 5 \), \( k = 3 \), and \( p = 7 \). Then \( |\Delta F_{5,3}(7)| = \binom{5}{3} + \binom{7}{3} \) and \( \Delta F_{5,3}(7) = \{12, 13, 23, 14, 24, 34, 15, 25, 35\} = [4]^2 \cup ([3]^2 \cup \{5\}) \).}

An immediate and easy consequence of Theorem 2.39 is the following corollary.

**Corollary 2.41.** Let \( k \in \mathbb{Z}^+ \). The new-shadows of the first \((k+1)\) \( k \)-sets in squashed order have cardinalities

\[ k, k - 1, \ldots, 1, 0 \]

respectively. The new-shades of the last \((n - k + 1)\) \( k \)-subsets of \([n]\) in squashed order have cardinalities

\[ n - k, n - k - 1, \ldots, 1, 0 \]

respectively.

**Proof.** The \( k \)-binomial representation of 1 is \( \binom{k}{k} \). The \( k \)-binomial representation of 2 is \( \binom{k}{k} + \binom{k-1}{k-1} \). More generally, for \( p \in \mathbb{Z}^+ \) and \( 1 \leq p \leq k \), the \( k \)-binomial representation of \( p \) is \( \sum_{i=k-p+1}^{k} \binom{i}{k} \). The \( k \)-binomial representation of \( k + 1 \) is \( \binom{k+1}{k} \). The result for the new-shadows of the first \((k+1)\) \( k \)-sets in squashed order follows from Theorem 2.39 and the definition of the new-shadow. The sizes of the new-shades of the last \((n - k + 1)\) \( k \)-subsets of \([n]\) in squashed order, that is, of the first \((n - k + 1)\) \( k \)-subsets of \([n]\) in antilexicographic order, are obtained from the sizes of the new-shadows of the first \((n - k + 1)\) \((n - k)\)-sets in squashed order by applying Lemma 2.20 and Observation 2.4. \(\square\)

The next three observations derive from Corollary 2.41.

**Observation 2.42.** Let \( n, k \in \mathbb{Z}^+ \) be such that \( 0 < k < n \). Let \( m, p \in \mathbb{Z}^+ \).

(i) Assume that \( 0 < m \leq p < k \). Then \( |\Delta_N L(m, F_{n,k}(p))| \geq 2m \).

(ii) Assume that \( 0 < m \leq p < n - k \). Then \( |\nabla_N F(m, L_{n,k}(p))| \geq 2m \).

Note that, for \( p \leq n - k + 1 \), \( |\nabla L_{n,k}(p)| = \frac{(n-k)(n-k+1)}{2} - \frac{(n-k-p)(n-k-p+1)}{2} = (n - k - p)p + \frac{p(p+1)}{2} \) by Corollary 2.41. Thus
Observeation 2.43. Let \( n, k, p \in \mathbb{Z}^+ \) be such that \( p \leq n - k + 1 \). Then \( |\nabla L_{n,k}(p)| = \frac{2n-2k+1-n}{2} \times p \).

Observeation 2.44. Let \( k \in \mathbb{N} \) and let \( A \) be an antichain on \([n]\) with parameters \( p_i \).

(i) Let \(|I^{(k)}A| < k + 1 \). Then \( p_i = 0 \) for \( i > k \).

(ii) Let \(|X^{(k)}A| < n - k + 1 \). Then \( p_i = 0 \) for \( i < k \).

Although Theorem 2.39 gives the exact value for \( |\Delta F_{n,k}(p)| \), this result is not very convenient to use in an analytical sense as it involves finding the \( k \)-binomial representation of \( p \). The following theorem can in some instances be more convenient to use. It is an equivalent formulation of Theorem 2.39 expressed in terms of the real \( k \)-binomial representation of \( p \).

Theorem 2.45 (Lovász [21, pp. 81 & 459]). Let \( \mathcal{B} \) be a collection of \( k \)-subsets of \([n]\) such that \( |\mathcal{B}| = p \). Assume that the real \( k \)-binomial representation of \( p \) is

\[
p = \binom{x}{k},
\]

\( x \in \mathbb{R}^+ \). Then

\[
|\Delta \mathcal{B}| \geq \binom{x}{k - 1}.
\]

The next theorem gives an upper bound for \( |\Delta F_{n,k}(p)| \). It is used to derive a lower bound for \( |\Delta_N L_{n,k}(p)| \), this is done in Section 2.8 (see Theorem 2.59).

Theorem 2.46 (Maire [22]). Let \( p \in \mathbb{N} \) be such that \( p \leq \binom{n}{k} \). Then

\[
|\Delta F_{n,k}(p)| \leq kp - p(p - 1) \frac{k(n - k)}{(\binom{n}{k} - 1)(n - k + 1)}.
\]

Equality holds when \( p = 0 \) or \( p = \binom{n}{k} \).

Using Theorem 2.46 and applying Lemma 2.18 it is possible to derive an upper bound for the size of the collection \( \nabla L_{n,k}(p) \). This last result is not given here.
2.8 Bounds for New-Shadows and New-Shades

In the previous section Sperner’s lemma and Theorem 2.39 gave lower bounds for \(|\triangle F_{n,k}(p)|\) in terms of \(p\) and of the \(k\)-binomial representation of \(p\) respectively. Here we give lower bounds and upper bounds for \(|\triangle F_{n,k}(p)|\) which are expressed in terms of the size of the shadow or of the new-shadow of appropriately chosen collections of sets. Moreover, bounds for the size of the new-shadow of a collection of \(p\) consecutive \(k\)-subsets of \([n]\) in squashed order are also given. These results are due to Clements [8] and are stated in Theorems 2.47 and 2.48. It will be shown in Chapter 3 that Theorems 2.47 and 2.48 can both be strengthened.

**Theorem 2.47 (Clements [8]).** Let \(p \in N\) be such that \(p \leq \binom{n}{k}\). Then

\[ |\triangle F_{n,k}(p)| \geq |\triangle NC_{n,k}(p)| \geq |\triangle NL_{n,k}(p)|. \]

**Theorem 2.48 (Clements [8]).** Let \(p \in N\) be such that \(p \leq \min\{\binom{n}{k}, \binom{n}{k+1}\}\). Then

\[ |\triangle F_{n,k}(p)| \leq |\triangle F_{n,k+1}(p)|, \text{ and} \]

\[ |\triangle NL_{n,k}(p)| \leq |\triangle NL_{n,k+1}(p)|. \]

A similar result can be obtained for shades and new-shades as shown by Corollaries 2.49 and 2.50 below. The corollaries are respectively obtained by replacing \(k\) by \(n-k\) in Theorem 2.47 or \(n-k-1\) in Theorem 2.48 and applying Lemma 2.21.(ii) and Observations 2.9 and 2.3.

**Corollary 2.49.** Let \(p \in N\) be such that \(p \leq \binom{n}{k}\). Then

\[ |\nabla L_{n,k}(p)| \geq |\nabla NC_{n,k}(p)| \geq |\nabla NL_{n,k}(p)|. \]

**Corollary 2.50.** Let \(p \in N\) be such that \(p \leq \min\{\binom{n}{k}, \binom{n}{k+1}\}\). Then

\[ |\nabla L_{n,k}(p)| \geq |\nabla L_{n,k+1}(p)|, \text{ and} \]

\[ |\nabla NL_{n,k}(p)| \geq |\nabla NL_{n,k+1}(p)|. \]

**Example 2.51.** \( |\triangle F_{5,3}(2)| = |\triangle \{123, 124\}| = 5 \geq |\triangle NC_{5,3}(2)| = |\triangle \{125, 135\}| = 3 \geq |\triangle NL_{5,3}(2)| = |\triangle \{245, 345\}| = 0. \)

\( |\triangle F_{5,3}(2)| = |\triangle \{123, 124\}| = 5 \geq |\triangle F_{5,2}(2)| = |\triangle \{12, 13\}| = 3. \)
Note 2.52.

Let \( m \in \mathbb{N} \). In Theorem 2.47 and Corollary 2.49, the collection \( C_{n,k}(p) \) denotes any collection of \( p \) consecutive \( k \)-subsets of \([n]\) in squashed order. In particular, for any \( m \in \mathbb{Z}^+ \), the collection \( C_{n,k}(p) \) can be replaced by the collection \( N_{n,k}^{m}(p) \) which comes immediately after the first \( m \) \( k \)-sets in squashed order so that \( |\triangle F_{n,k}(p)| \geq \left| \Delta_{N} N_{n,k}^{m}(p) \right| \geq |\Delta_{N} L_{n,k}(p)| \) and \( \nabla F_{n,k}(p) \geq \left| \nabla_{N} N_{n,k}^{m}(p) \right| \geq \left| \nabla_{N} F_{n,k}(p) \right| \).

Similarly, the collection \( C_{n,k}(p) \) can be replaced by the collection \( P_{n,k}^{m}(p) \) which comes immediately before the last \( m \) \( k \)-sets in squashed order so that \( |\triangle F_{n,k}(p)| \geq \left| \Delta_{N} P_{n,k}^{m}(p) \right| \geq |\Delta_{N} L_{n,k}(p)| \) and \( \nabla F_{n,k}(p) \geq \left| \nabla_{N} P_{n,k}^{m}(p) \right| \geq \left| \nabla_{N} F_{n,k}(p) \right| \).

There are several more corollaries of Theorem 2.47. Corollary 2.53 shows that \( \Delta F_{n,k}(p) \) is subadditive and Corollaries 2.54 and 2.55 derive further inequalities from Theorem 2.47.

Corollary 2.53 (Clements [6]). Let \( p_1, p_2 \in \mathbb{N} \) be such that \( p_1 + p_2 \leq \binom{n}{k} \). Then

\[
|\triangle F_{n,k}(p_1 + p_2)| \leq |\triangle F_{n,k}(p_1)| + |\triangle F_{n,k}(p_2)|.
\]

Proof.

\[
|\triangle F_{n,k}(p_1 + p_2)| = |\triangle F_{n,k}(p_1)| + |\triangle_{N} N_{n,k}^{p_1}(p_2)|
\]

(by Observation 2.11)

\[
\leq |\triangle F_{n,k}(p_1)| + |\triangle F_{n,k}(p_2)|
\]

(by Note 2.52 and Theorem 2.47)

as required. \( \square \)

Corollary 2.54. Let \( p_1, p_2 \in \mathbb{N} \) be such that \( p_2 \leq p_1 \leq \binom{n}{k} \). Then

\[
|\triangle F_{n,k}(p_1 - p_2)| \leq |\triangle F_{n,k}(p_1)| - |\triangle_{N} L_{n,k}(p_2)|.
\]

Proof. By Note 2.52 and Theorem 2.47, \( |\triangle F_{n,k}(p_1)| - |\triangle_{N} L_{n,k}(p_2)| \geq |\triangle F_{n,k}(p_1)| - |\triangle_{N} N_{n,k}^{p_1-p_2}(p_2)|. \) Note that \( |\triangle F_{n,k}(p_1)| - |\triangle_{N} N_{n,k}^{p_1-p_2}(p_2)| = |\triangle F_{n,k}(p_1 - p_2)|. \) This proves the corollary. \( \square \)
Corollary 2.55. Let $p_1, p_2 \in \mathbb{N}$ be such that $p_1 + p_2 \leq \binom{n}{k}$. Then

$$|\Delta_{N}L_{n,k}(p_1 + p_2)| \leq |\Delta_{N}F_{n,k}(p_1)| + |\Delta_{N}L_{n,k}(p_2)|.$$ 

Proof. By Note 2.52 and Theorem 2.47, $|\Delta_{N}F_{n,k}(p_1)| \geq |\Delta_{N}F_{n,k}^{p_2}(p_1)|$. By Observation 2.11, $|\Delta_{N}L_{n,k}(p_1 + p_2)| = |\Delta_{N}F_{n,k}^{p_2}(p_1)| + |\Delta_{N}L_{n,k}(p_2)| \leq |\Delta_{N}F_{n,k}(p_1)| + |\Delta_{N}L_{n,k}(p_2)|$ as required.

The remainder of this section involves finding two further lower bounds for the size of new-shadows. To do so, we begin by giving a lower bound for $|\nabla_{N}\mathcal{B}|$ where $\mathcal{B}$ is a collection of consecutive $k$-sets in squashed order, in the manner of Theorem 2.39.

\diamond \textbf{Theorem 2.56.} Let $\mathcal{B}$ be a collection of $p$ consecutive $k$-subsets of $[n]$ in squashed order. Assume that the $k$-binomial representation of $p$ is

$$p = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}.$$ 

Then

$$|\nabla_{N}\mathcal{B}| \geq \binom{a_k}{k+1} + \binom{a_{k-1}}{k} + \ldots + \binom{a_t}{t+1}.$$ 

Equality holds when $\mathcal{B}$ is an initial segment of $k$-sets in squashed order.

Proof. The proof is very similar to that of Theorem 2.39. Let $p = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}$ be the $k$-binomial representation of $p$. By Lemma 2.15, $F_{n,k}(p)$ is the union of the non-intersecting collections $[a_k]^k$, $[a_{k-1}]^{k-2}\{a_{k-1}+1\}$, $[a_{k-2}]^{k-2}\{a_{k-2}+1, a_{k-1}+1\}$, $\ldots$, $[a_t]^t\{a_{t+1}+1, \ldots, a_k+1\}$. Denote the collection $[a_k]^k$ by $\mathcal{B}_k$, and the collections $[a_i]^i \{a_{i+1}+1, \ldots, a_k+1\}$, $i = k-1, \ldots, t$, by $\mathcal{B}_i$. Note that $|\nabla_{N}F_{n,k}(p)| = \sum_{i=t}^{k} |\nabla_{N}\mathcal{B}_i|$ by Observation 2.11.

By Lemma 2.30, $|\nabla_{N}\mathcal{B}_i| = |\nabla_{N}([a_i]^i)|$ for $i = t, \ldots, k-1$. Since $[a_i]^i = L_{a_i,i}(\binom{a_i}{i})$ by Observation 2.7, it follows that $|\nabla_{N}F_{n,k}(p)| = \sum_{i=t}^{k} |\nabla_{N}([a_i]^i)| = \sum_{i=t}^{k} \binom{a_i}{i+1}$ by Observations 2.9 and 2.8. Theorem 2.56 now follows from Corollary 2.49.

\textbf{Example 2.57.} As in Examples 2.16 and 2.40 let $n = 5$, $k = 3$, and $p = 7$. Then $|\nabla_{N}F_{5,3}(7)| = \binom{4}{4} + \binom{3}{3}$ and $\nabla_{N}F_{5,3}(7) = \{1234, 1235\} = [4]^4 \cup ([3]^3 \cup \{5\})$. \diamond
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It is now possible to derive a lower bound for $|\Delta_{N}B|$ where $B$ is a collection of consecutive $k$-sets in squashed order, by applying Lemma 2.22.

\[ \diamond \textbf{Corollary 2.58.} \text{Let } B \text{ be a collection of } p \text{ consecutive } k\text{-subsets of } [n] \text{ in squashed order. Assume that the } (n-k)\text{-binomial representation of } p \text{ is} \]

\[ p = \binom{a_{n-k}}{n-k} + \binom{a_{n-k-1}}{n-k-1} + \ldots + \binom{a_{t}}{t}. \]

Then

\[ |\Delta_{N}B| \geq \binom{a_{n-k}}{n-k + 1} + \binom{a_{n-k-1}}{n-k} + \ldots + \binom{a_{t}}{t+1}. \]

Equality holds when $B$ is an initial segment of $k\text{-subsets of } [n]$ in antilexicographic order.

\[ \text{Proof.} \text{ By Lemma 2.22, } |\Delta_{N}L_{n,k}(p)| = |\nabla_{N}F_{n,n-k}(p)|. \text{ The result then follows from Theorems 2.56 and 2.47 and Observation 2.2.} \]

The last theorem in this section gives a lower bound for $|\Delta_{N}L_{n,k}(p)|$ by applying Theorem 2.46.

\[ \diamond \textbf{Theorem 2.59 \ (with Branković).} \text{Let } p \in N \text{ be such that } p \leq \binom{n}{k}. \text{ Then} \]

\[ |\Delta_{N}L_{n,k}(p)| \geq \binom{n}{k - 1} - k\binom{n}{k} - p + \binom{n}{k} - p - 1 + \frac{k(n-k)}{(\binom{n}{k} - 1)(n-k+1)}. \]

Equality holds when $p = 0$ or $p = \binom{n}{k}$. \]

\[ \text{Proof.} \text{ Note that } |\Delta_{N}L_{n,k}(p)| = \binom{n}{k - 1} - |\Delta_{F_{n,k}}(\binom{n}{k} - p)| \text{ by definition of the new-shadow. Further, } |\Delta_{F_{n,k}}(\binom{n}{k} - p)| \leq k(\binom{n}{k} - p) - (\binom{n}{k} - p)(\binom{n}{k} - p - 1) \frac{k(n-k)}{(\binom{n}{k} - 1)(n-k+1)} \text{ by Theorem 2.46. The result follows.} \]

We have stated that the exact value of $|\Delta_{F_{n,k}}(p)|$ given by Theorem 2.39 is not convenient to use in an analytical sense. Similarly, the exact value given by Corollary 2.58 or the lower bound given by Theorem 2.59 for $|\Delta_{N}L_{n,k}(p)|$ are also not convenient to use. This is one of the reasons why Theorem 7.1 (see Chapter 7) is interesting.
Although Theorem 7.1 does not give a lower bound for $|\triangle_N L_{n,k}(p)|$, it does give a simple lower bound for the sum $|\triangle_N L_{n,k+1}(p)| + |\triangle_N L_{n,k-1}(p)|$. As seen in Chapter 8, Theorem 7.1 enables us to solve the flat antichain conjecture in several cases.

2.9 New-Shadow and New-Shade of a Set

In this section we determine $|\triangle_N B|$ and $|\nabla_N B|$ for any set $B$. Lemmas 2.60 and 2.62 are two alternative versions of the same result. Lemmas 2.60 and 2.62 are required for the algorithm which is used to prove part of Theorem 7.1 (see Appendix D).

\begin{lemma}
Let $B = \{b_1, b_2, \ldots, b_k\}$, $b_1 < b_2 < \ldots < b_k$, be a $k$-subset of $[n]$, $k \in \mathbb{Z}^+$. Let $l$ be the smallest $i$ for which $b_i > i$ and let $l = k + 1$ whenever $b_i = i$ for all $i \in [k]$. Let $j$ be the smallest element in $B$. Then

(i) $|\triangle_N B| = l - 1$,

(ii) $|\nabla_N B| = j - 1$.

\end{lemma}

\textit{Proof.} Let $B$, $j$, and $l$ be as in the statement of the lemma. We prove (ii) first and then derive (i) from (ii).

(ii) Let $m \in \mathbb{Z}^+$, $m \notin B$, and $D = B \cup \{m\}$. We consider two cases.

(a) $m > j$. Let $C \subset D$ be the $k$-set $(B \setminus \{j\}) \cup \{m\}$. Then $C \subseteq B$ and it follows that $D \notin \nabla_N B$ as $D \supset C$ and $D \supset B$.

(b) $m < j$. Let $C$ be any $k$-subset of $D$ such that $C \neq B$. Then $C = (B \setminus \{r\}) \cup \{m\}$ for some $r \in B$. It follows that $C \supseteq B = \{m, r\}$ where $m < j \leq r$. Thus $D$ is such that $D \supset C$, $D \supset B$ and $C \subseteq B$. This holds for all subsets $C$ of $D$. Therefore $D \in \nabla_N B$. There are exactly $j - 1$ supersets $D$ of $B$ of the form $B \cup \{m\}$, $m < j$. Hence $|\nabla_N B| = j - 1$ as required.

This proves (ii).

(i) We consider two cases.
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(a) Assume that \( l \neq k + 1 \), and consider the complement \( B' \) of \( B \). Note that \( B' \) always contains a smallest element since \( B' \neq \emptyset \) when \( l \neq k + 1 \). The smallest element in \( B' \) is \( l \). It follows that \( |\triangle_N B| = |\nabla_N B| = l - 1 \) by Lemma 2.21.(i) and by (ii).

(b) Assume now that \( l = k + 1 \). Then \( B = [k] \) and \( |\triangle_N B| = k = l - 1 \) by Lemma 2.41. This concludes the proof of Lemma 2.60. \( \square \)

Example 2.61. For \( B = \{1345\} \), \( |\triangle_N B| = 1 \) and \( |\nabla_N B| = 0 \). For \( B = \{1256\} \), \( |\triangle_N B| = 2 \) and \( |\nabla_N B| = 0 \). For \( B = \{2489\} \), \( |\triangle_N B| = 0 \) and \( |\nabla_N B| = 1 \). \( \bullet \)

Lemma 2.62. Let \( p \in \mathbb{Z}^+ \) and let \( \sum_{i=1}^{k}(\binom{t}{i}) \) be the k-binomial representation of \( p \). Let \( B \) be the \((p+1)th\) \( k \)-set in squashed order. Then

(i) \( |\triangle_N B| = t - 1 \),

\( \diamond \) (ii) \( |\nabla_N B| = \begin{cases} a_1 & \text{for } t = 1, \\ 0 & \text{for } t > 1, \end{cases} \)

Proof. (i) is a result stated in [7]. To prove (ii) two cases are considered.

(a) Assume that \( t = 1 \) and let \( \sum_{i=1}^{k}(a_i) \) be the \( k \)-binomial representation of \( p \). By Lemma 2.15, \( F_{n,k}(p) \) is the union of the non-intersecting collections \( [a_k]^k, [a_{k-1}]^{(k-1)}\cup\{a_k + 1\}, \ldots, [a_1]^1\cup\{a_2 + 1, \ldots, a_k + 1\} \). The last set in squashed order of \( F_{n,k}(p) \) is therefore the set \( \{a_1, a_2 + 1, \ldots, a_k + 1\} \). \( B \) is the \((p+1)th\) set in squashed order, whence \( B = \{a_1 + 1, a_2 + 1, \ldots, a_k + 1\} \) as \( a_1 < a_2 \) by the definition of the \( k \)-binomial representation of a number. By Lemma 2.60.(ii) it follows that \( |\nabla_N B| = a_1 \) as required.

(b) Assume that \( t > 1 \). Let \( B = \{b_1, b_2, \ldots, b_k\} \), \( b_1 < b_2 < \ldots < b_k \), and let \( b_{k+1} = k + 2 \). Let \( l \) be the smallest \( i \) for which \( b_i > i \). \( |\triangle_N B| = t - 1 \geq 1 \) by (i) and \( |\triangle_N B| = l - 1 \) by Lemma 2.60.(i); thus \( l > 1 \) and \( b_1 = 1 \). By Lemma 2.60.(ii) it follows that \( |\nabla_N B| = 0 \) as required. \( \square \)

The next lemma follows from Lemmas 2.60 and 2.62. It highlights one relationship between \( |\triangle_N B| \) and \( |\nabla_N B| \).
\textbf{Lemma 2.63.} Let $B$ be a set. The two following statements are equivalent:

(i) $|\Delta_N B| = 0$ if and only if $1 \notin B$ and $|\nabla_N B| = 0$ if and only if $1 \in B$.

(ii) $|\Delta_N B| > 0$ if and only if $|\nabla_N B| = 0$.

\section{Relationships between $\Delta_N(\nabla_N B)$ and $B$ and between $\nabla_N(\Delta_N B)$ and $B$}

We begin by establishing a relationship between $\Delta_N(\nabla_N B)$ and $B$ where $B$ is a collection of consecutive sets of the same size in squashed order.

\textbf{Lemma 2.64.} Let $a \in N$ and $B = L(p, \Delta F_{n,k+1}(a))$ for $p \leq |\Delta F_{n,k+1}(a)|$. Then $\Delta_N(\nabla_N B \cap F_{n,k+1}(a)) \supseteq B$.

\textit{Proof.} Let $a \in N$, $A_1 = F_{n,k+1}(a)$, and $A_2 = N_{n,k}^{\Delta A_1}(\binom{n}{k} - |\Delta A_1|)$. Then $A_1 \cup A_2$ is a squashed and full antichain. Let $B$ be as in the statement of the lemma and let $C = \nabla_N B \cap A_1$. We prove that $\Delta_N C \supseteq B$. The proof requires the following three claims.

\textbf{Claim 1.} The collections $C$ and $\Delta_N C$ are collections of consecutive $(k+1)$-sets and $k$-sets respectively in squashed order.

\textit{Proof of Claim 1.} This is easily seen by Lemma 2.26.

\textbf{Claim 2.} $C$ is the collection of the last $|C|$ $(k+1)$-sets of $A_1$.

\textit{Proof of Claim 2.} Assume that $A$ is a set such that $A \in A_1$ and $A \succ_S C$. Thus $A \notin C$ and $A \notin \nabla_N B$. By Lemma 2.25, there exists a set $B \succ_S B$ which is such that $A \in \nabla_N B$. However $B \succ_S B$ implies that $B \in A_2$, contradicting the fact that $A_1 \cup A_2$ is an antichain. Therefore the last set in $C$ is the last set in $A_1$. This, together with Claim 1, proves Claim 2.

\textbf{Claim 3.} $\Delta_N C$ is the collection of the last $|\Delta_N C|$ $k$-sets of $\Delta A_1$. 
Proof of Claim 3. First note that $\triangle A_1$ is a collection of $k$-sets in squashed order by Lemma 2.26 and by the definition of $A_1$. Let $A$ be any $k$-set such that $A >_S \triangle_N C$. By Claim 2, $C$ is the collection of the last $|C|$ $(k+1)$-sets of $A_1$. Thus $A \in A_2$ as $A_1 \cup A_2$ is a squashed antichain. This, together with Claim 1, proves Claim 3.

Assume that $\triangle_N C \not\subseteq B$. That is, there exists a set $B \in B$ such that $B \notin \triangle_N C$. By Claim 3, and as $B = L(p, \triangle A_1)$, $B$ is such that $B <_S \triangle_N C$. Choose $B$ to be the set which immediately precedes the sets in $\triangle_N C$ in squashed order. There exists a $(k+1)$-set $A$ such that $B \in \triangle_N A$ and $A \notin C$. Then $A <_S C$ by Lemma 2.25.

That is, $A \in A_1$, $A \notin C$, and $A \notin \nabla_N B$ by the definition of $C$. Thus $A \notin \nabla_N B$ as $B \in B$. However, $B \subseteq A$. This implies that there exists a $k$-set $D$, $D >_S B$, which is such that $A \in \nabla_N D$. As $A \notin \nabla_N B$, $D >_S B$ and $D \in A_2$. This contradicts the fact that $A_1 \cup A_2$ is an antichain. Thus $\triangle_N C \supseteq B$ as required. \hfill \Box

Example 2.65. Let $n = 5$, $k = 2$, $a = 4$, and $p = 2$. Then $F_{5,3}(4) = \{123, 124, 134, 234\}$ and $B = \{24, 34\}$. It follows that $\nabla_N B = \{124, 134, 234\} = \nabla_N B \cap F_{5,3}(4)$ so $\triangle_N (\nabla_N B \cap F_{5,3}(4)) = \{14, 24, 34\} \supseteq B$.

From Lemma 2.64 it follows that

\[ |\triangle_N L_{n,k+1}(|\nabla L_{n,k}(p)|)| \geq p. \]

\[ \text{Proof.} \quad \text{In Lemma 2.64 take } a = \binom{n}{k+1} \text{ so that } B = L_{n,k}(a) \text{ and } \nabla_N B \subseteq F_{n,k+1}(a). \text{ Therefore } \nabla_N B \cap F_{n,k+1}(a) = \nabla_N B. \text{ It follows that } |\triangle_N(\nabla_N B)| \geq |B| \text{ by taking cardinalities in Lemma 2.64. Note that } \nabla_N B = \nabla L_{n,k}(p) \text{ by Observation 2.9 as } B = L_{n,k}(p), \text{ and that } \nabla L_{n,k}(p) = L_{n,k+1}(|\nabla L_{n,k}(p)|) \text{ by Lemma 2.23. Thus } |\triangle_N L_{n,k+1}(|\nabla L_{n,k}(p)|)| \geq p \text{ as required.} \hfill \Box \]

Combining Lemmas 2.66 and 2.21(ii) we obtain

\[ |\nabla_N F_{n,k}(|\triangle F_{n,k+1}(p)|)| \geq p. \]

\[ \text{\diamond \ Lemma 2.67. \ Let } p \in N \text{ with } p \leq \binom{n}{k}. \text{ Then} \]

\[ |\nabla_N F_{n,k}(|\triangle F_{n,k+1}(p)|)| \geq p. \]
Proof. We have

\[ p \leq |\Delta_{N} L_{n,n-k}(\nabla L_{n,n-k-1}(p))| \]

(by Lemma 2.66)

\[ = \left| \nabla_{N} \left( L_{n,n-k}(\nabla L_{n,n-k-1}(p)) \right)' \right| \]

(by Lemma 2.21 (ii))

\[ = |\nabla_{N} F_{n,k}(\nabla L_{n,n-k-1}(p))| \]

(by Observation 2.3)

\[ = |\nabla_{N} F_{n,k}(\Delta F_{n,k+1}(p))| \]

(by Lemma 2.18)

as required.

Next we establish a relationship between $\nabla_{N}(\Delta_{N}B)$ and $B$ where $B$ is a collection of consecutive sets of the same size in squashed order. This is done in Lemma 2.68 below. The proof of Lemma 2.68 is similar to that of Lemma 2.64 so some details are omitted.

\begin{itemize}
  \item \textbf{Lemma 2.68.} Let $a \in \mathbb{N}$ and $B = L(p, F_{n,k+1}(a))$ for $p \leq [F_{n,k+1}(a)]$. Then
  \[ (\nabla_{N}(\Delta_{N}B)) \cap F_{n,k+1}(a) \subseteq B. \]
\end{itemize}

Proof. Let $a \in \mathbb{N}$, $A_{1} = F_{n,k+1}(a)$, and $A_{2} = N_{n,k}^{\Delta A_{1}} \binom{n}{k} - |\Delta A_{1}|$.

Then $A_{1} \cup A_{2}$ is a squashed and full antichain. Let $B$ be as in the statement of the lemma and let $C = \Delta_{N}B$. We prove that $\nabla_{N}C \cap A_{i} \subseteq B$. The proof requires the following three claims.

\begin{itemize}
  \item \textbf{Claim 1.} The collections $C$ and $\nabla_{N}C \cap A_{1}$ are collections of consecutive $k$-sets and $(k + 1)$-sets respectively in squashed order.
  \item \textbf{Claim 2.} $C$ is the collection of the last $|C|$ $k$-sets of $\Delta A_{1}$.
  \item \textbf{Claim 3.} $(\nabla_{N}C \cap A_{1})$ is the collection of the last $|\nabla_{N}C \cap A_{1}| (k + 1)$-sets of $A_{1}$.
\end{itemize}
The proofs of Claims 1, 2, and 3 are very similar to the proofs of Claims 1, 3, and 2 respectively in the proof of Lemma 2.64 and they are not given. Although only Claim 3 is required in the argument which follows we stated Claims 1 and 2 for clarity purpose since they are needed to prove Claim 3.

Assume that \((\triangle_N C \cap A_1) \not\subseteq B\). That is, there exists a set \(B \in (\triangle_N C \cap A_1)\) such that \(B \not\subseteq B\). By Claim 3, and as \(B = L(p, A_1)\), \(B\) is such that \(B <_S B\). Choose \(B\) to be the set which immediately precedes the sets in \(B\) in squashed order. As \(B \in (\triangle_N C \cap A_1)\) there exists a \(k\)-set \(A\) such that \(B \subseteq \triangle_N A\) and \(A \in C\). Since \(A \in C\) and \(C = \triangle_N B\) there exists a set \(D \in B\) such that \(A \subseteq \triangle_N D\). Therefore we have that \(A \subseteq B\), \(A \subseteq D\), \(B <_S D\) as \(B <_S B\). Thus \(A \in \triangle_N B\) and \(A \not\subseteq \triangle_N D\). This is a contradiction and we conclude that \((\triangle_N C \cap A_1) \subseteq B\).

\[\Box\]

**Example 2.69.** Let \(n = 5\), \(k = 2\), \(a = 6\), and \(p = 3\). Then \(F_{5,3}(6) = \{123, 124, 134, 234, 125, 135\}\) and \(B = \{234, 125, 135\}\). It follows that \(\triangle_N B = \{15, 25, 35\}\) and \(\triangle_N (\triangle_N B) = \{125, 135, 235\}\) so \((\triangle_N (\triangle_N B)) \cap F_{5,3}(6) = \{125, 135\} \subseteq B\).

From Lemma 2.68 it follows that

\[\Diamond \text{ Lemma 2.70.} \text{ Let } p \in N \text{ with } p \leq \binom{n}{k}. \text{ Then} \]

\[|\nabla L_n, k(|\triangle_N L_{n, k+1}(p)|)| \leq p.\]

**Proof.** In Lemma 2.68 take \(a = \binom{n}{k+1}\) so that \(B = L_{n, k+1}(p)\) and \(\nabla (\triangle_N B) \subseteq F_{n, k+1}(a)\). Also, \(\triangle_N B = L_{n, k}(|\triangle_N L_{n, k+1}(p)|)\) by Lemma 2.27. The remainder of the proof is very similar to the proof of Lemma 2.66 and is not given. \[\Box\]

Combining Lemmas 2.70 and 2.21(ii) one obtains

\[\Diamond \text{ Lemma 2.71.} \text{ Let } p \in N \text{ with } p \leq \binom{n}{k}. \text{ Then} \]

\[|\triangle F_{n, k+1}(|\nabla N F_{n, k}(p)|)| \leq p.\]

**Proof.** The proof is very similar to the proof of Lemma 2.67 and is not given. \[\Box\]
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**Note 2.72.**

In [13, p. 130] the following statement, expressed in terms of multisets in lexicographic order, is made: \(|\triangle_N L_{n,k} (b)| = a\) if and only if \(|\nabla L_{n,k-1} (a)| = b^*\). This statement is wrong when applied to the special case of sets.

If \(k\) is replaced by \(n - k\) and Lemmas 2.22 and 2.18 are applied, then saying that \(|\triangle_N L_{n,k} (b)| = a\) if and only if \(|\nabla L_{n,k-1} (a)| = b^*\) is equivalent to saying that \(|\nabla F_{n,k} (b)| = a\) if and only if \(|\triangle F_{n,k} (a)| = b^*\). This last statement implies that the two equalities \(|\nabla F_{n,k} (|\triangle F_{n,k+1} (a)| = a\) and \(|\triangle F_{n,k+1} (|\nabla F_{n,k} (b)| = b^*\) hold.

However there are cases where these equalities do not hold as the following example demonstrates.

**Example 2.73.** Let \(n = 4\) and \(k = 2\). Then \(|\nabla_N F_{4,2} (|\triangle F_{4,3} (3)|)| = |\nabla_N F_{4,2} (|\triangle \{123, 124, 134\}|)| = |\nabla \{12, 13, 23, 24\}| = |\{123, 124, 134, 234\}| = 4\). Also, \(|\triangle F_{4,3} (|\nabla N F_{4,2} (4)|)| = |\triangle F_{4,3} (|\nabla \{12, 13, 23, 14\}|)| = |\triangle \{123\}| = |\{12, 13, 23\}| = 3\).

---

**2.11 Antichains**

Three major results on antichains are Theorem 2.74 (Sperner’s theorem), the LYM inequality (Theorem 2.75), and Theorem 2.77 below. Theorem 2.74 gives an upper bound for the size of an antichain, and the LYM inequality states a necessary condition for the parameters of a collection of sets to be the parameters of an antichain. Theorem 2.77 states necessary and sufficient conditions for the parameters of a collection of sets to be the parameters of an antichain.

**Theorem 2.74 (Sperner [30]).** Let \(\mathcal{A}\) be an antichain of subsets of \([n]\). Then

\[|\mathcal{A}| \leq \left( \begin{array}{c} n \\ \lfloor \frac{n}{2} \rfloor \end{array} \right).\]

Lubell, Yamamoto and Meschalkin generalised Theorem 2.74 as follows.
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Theorem 2.75 (The LYM inequality [1, p. 3]). Let $\mathcal{A}$ be an antichain on $[n]$ with parameters $p_0, \ldots, p_n$. Then
\[ \sum_{i=0}^{n} \frac{p_i}{\binom{n}{i}} \leq 1. \]

Equality in the LYM inequality is achieved in very specific cases only.

Theorem 2.76 (Bollobás [3, p. 13]). Equality is achieved in Theorem 2.75 if and only if $\mathcal{A}$ consists of all the $\binom{n}{k}$ $k$-sets, for some $k$.

Theorem 2.77 (Clements [5], Daykin et al. [11]). Let $p_0, p_1, \ldots, p_n$ be non-negative integers, and let $h$ and $l$ be the largest and smallest values of $i$ for which $p_i \neq 0$. Define the integers $n_h, \ldots, n_l$ by
\[

g_{h} = 0 \\
n_{h-1} = |\Delta F_{h}(p_h)| \\
n_{h-2} = |\Delta F_{h-1}(p_{h-1} + n_{h-1})| \\
\vdots \\
n_l = |\Delta F_{l+1}(p_{l+1} + n_{l+1})|.
\]

Then the integers $p_0, \ldots, p_n$ are the parameters of an antichain on $[n]$ if and only if
\[ p_i + n_i \leq \binom{n}{i} \]
for $l \leq i \leq h$.

Let the integers $p_i$, $0 \leq i \leq n$, and $n_i$, $l \leq i \leq h$, be as in Theorem 2.77. Let $\mathcal{B}$ be a collection of sets with parameters $p_i$. The $n_i$s are the sizes of the shadows on level $i$ of the squashed antichain with parameters $p_i$. By Theorem 2.36 the $n_i$s denote the minimum number of $i$-sets which cannot be in $\mathcal{B}$ if $\mathcal{B}$ is to be antichain. Therefore the condition $p_i + n_i \leq \binom{n}{i}$ for each $i = l, \ldots, h$ is a necessary condition for the $p_i$s to be the parameters of an antichain. It is also a sufficient condition as a squashed antichain with parameters $p_i$ can be constructed. Thus Theorem 2.77 can be restated as follows.
Theorem 2.78 ([1, p. 134]). There exists an antichain on \([n]\) with parameters \(p_0, \ldots, p_n\) if and only if there exists a squashed antichain with the same parameters.

By definition, a squashed collection of sets is an antichain. Sometimes the problem is to determine the universal set on which the squashed collection is defined. The following easily derived lemmas state a sufficient condition for a squashed flat counterpart of an antichain \(\mathcal{A}\) on \([n]\) to be an antichain on \([n]\). The first lemma, Lemma 2.79, states the condition in terms of the \(\mathcal{A}\)-projection of \(\mathcal{A}\) while the second lemma, Lemma 2.80, states the condition in terms of the \(\overline{\mathcal{A}}\)-image of \(\mathcal{A}\).

Lemma 2.79. Let \(\mathcal{A}\) be an antichain on \([n]\) and let \(\mathcal{A}^*\) be the squashed flat counterpart of \(\mathcal{A}\). Assume that \(|\Diamond(\overline{\mathcal{A}})\mathcal{A}^*| \leq |\Diamond(\overline{\mathcal{A}})\mathcal{A}|\). Then \(\mathcal{A}^*\) is an antichain on \([n]\).

Proof. Let \(\mathcal{A}\) and \(\mathcal{A}^*\) be as in the statement of the lemma and let \(k = |\overline{\mathcal{A}}|\). Although \(\mathcal{A}^*\) is an antichain, it remains to be proved that \(\mathcal{A}^*\) is an antichain on \([n]\). As \(\mathcal{A}\) is an antichain on \([n]\), \(\Diamond(k)\mathcal{A}\) is the disjoint union of the collections \(\Delta(k), \mathcal{A}(k), \) and \(\nabla(k)\mathcal{A}\), and \(|\Diamond(k)\mathcal{A}| \leq \binom{n}{k}\). Given the definition of \(k\) and \(\mathcal{A}^*\), \(\mathcal{A}^*\) consists of \(q_k\) \(k\)-sets, \(q_k > 0\), and possibly \(q_{k+1}\) \((k+1)\)-sets, and \(|\Diamond(k)\mathcal{A}^*| = |\Delta F_{n,k+1}(q_{k+1})| + q_k\). Therefore \(|\Delta F_{n,k+1}(q_{k+1})| + q_k \leq \binom{n}{k}\) since \(|\Diamond(k)\mathcal{A}^*| \leq |\Diamond(k)\mathcal{A}|\) by assumption. Note that \(q_{k+1} < \left(\begin{array}{c} n \\ k+1 \end{array}\right)\) as \(q_k > 0\) and \(|\Delta F_{n,k+1}(\left(\begin{array}{c} n \\ k+1 \end{array}\right))| = \binom{n}{k}\) by Observation 2.8. It follows that \(q_k\) and \(q_{k+1}\) are the parameters of an antichain on \([n]\) by Theorem 2.77.

Lemma 2.80. Let \(\mathcal{A}\) be an antichain on \([n]\) and let \(\mathcal{A}^*\) be the squashed flat counterpart of \(\mathcal{A}\). Assume that \(|\circ(\overline{\mathcal{A}})\mathcal{A}^*| \leq |\circ(\overline{\mathcal{A}})\mathcal{A}|\). Then \(\mathcal{A}^*\) is an antichain on \([n]\).

Proof. Let \(\mathcal{A}\) and \(\mathcal{A}^*\) be as in the statement of the lemma and let \(k = |\overline{\mathcal{A}}|\). Although \(\mathcal{A}^*\) is an antichain, it remains to be proved that \(\mathcal{A}^*\) is an antichain on \([n]\). Given the definition of \(k\) and \(\mathcal{A}^*\), we see that \(\circ(k)\mathcal{A}^* = \Diamond(k)\mathcal{A}^*\). Also, \(\circ(k)\mathcal{A} \subseteq \Diamond(k)\mathcal{A}\) since \(\nabla(k)\mathcal{A} \subseteq \nabla(k)\mathcal{A}\) by the definition of new-shade. Thus \(|\circ(k)\mathcal{A}^*| \leq |\circ(k)\mathcal{A}|\) implies that \(|\Diamond(k)\mathcal{A}^*| \leq |\Diamond(k)\mathcal{A}|\). Lemma 2.80 now follows from Lemma 2.79.
If $A$ is an antichain on $[n]$ with parameters $p$, then it is reasonable to ask what are the parameters of a flat counterpart of $A$. This is answered in the next lemma.

**Lemma 2.81.** Let $A$ be an antichain on $[n]$ with parameters $p$, and $|A| = k$. Let $h$ and $l$ respectively be the largest and smallest integer for which $p_i \neq 0$. Let $A^*$ be a flat counterpart of $A$ with parameters $q_i$, $i \geq 0$. Then

$$q_{k+1} = \sum_{i=1}^{h-k} ip_{k+i} - \sum_{i=1}^{k-1} ip_{k-i}$$

and

$$q_k = p_k + \sum_{i=1}^{k-1} (i+1)p_{k-i} - \sum_{i=2}^{h-k} (i-1)p_{k+i}.$$  

**Proof.** Let $A$, $k$, $h$, $l$, and $A^*$ be as in the statement of the lemma. Let $V(A) = k|A|+r$, $0 \leq r < |A|$. Then $q_{k+1} = r$, $q_k = |A|-r$, and $q_i = 0$ for $i \neq k, k+1$. Note that $V(A) = \sum_{i=1}^{h-k} (k+i)p_{k+i} + kp_k + \sum_{i=1}^{k-1} (k-i)p_{k-i} = k|A| + \sum_{i=1}^{h-k} ip_{k+i} - \sum_{i=1}^{k-1} ip_{k-i}$. Therefore $r = \sum_{i=1}^{h-k} ip_{k+i} - \sum_{i=1}^{k-1} ip_{k-i}$ and the lemma follows. \hfill \Box

Let $A$ and $B$ be two subsets of $[n]$ such that $A \not\subseteq B$. Then $B' \not\subseteq A'$. This means that the complement of an antichain $A$ on $[n]$ is also an antichain on $[n]$. The next three observations follow from this.

**Observation 2.82.** Let $A \in \Lambda_{n,s}$. Then $A' \in \Lambda_{n,s}$, $V(A') = n|A| - V(A)$, and $\overline{A'} = n - \overline{A}$.

**Observation 2.83.** If $A$ is a flat antichain on $[n]$ then $A'$ is also a flat antichain on $[n]$.

**Observation 2.84.** Let $A \in \Lambda_{n,s}$ and let $A^*$ be an antichain on $[n]$ which is a flat counterpart of $A$. Let $A'$ and $A'^*$ be the complements of $A$ and $A^*$ respectively. Then $A'^*$ is an antichain on $[n]$ which is a flat counterpart of $A'$.

**Example 2.85.** Let $n = 5$, $A = \{1234, 125, 35, 45\}$ and $A^* = \{123, 124, 134, 15\}$. Then $A' = \{123, 124, 34, 5\}$ and $A'^* = \{234, 25, 35, 45\}$.

The section concludes with the following generalisation of Observation 2.2 which can be seen as providing an alternative definition of an antilexicographic antichain on $[n]$. 

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Lemma 2.86. $\mathcal{A}$ is a squashed antichain on $[n]$ if and only if $\mathcal{A}'$ is an antilexicographic antichain on $[n]$.

Proof. The proof is partly adapted from [1, p. 140]. Assume that $\mathcal{A}$ is a squashed antichain on $[n]$ with parameters $p_i$ and let $h$ and $l$ respectively be the largest and smallest element for which $p_i \neq 0$. Recall that by definition of a squashed antichain the sets in $\triangle^{(i)} \mathcal{A}$, $l \leq i \leq h$, precede the sets in $\mathcal{A}^{(i)}$ so that the collection $\triangle^{(i)} \mathcal{A} \cup \mathcal{A}^{(i)}$ is an initial segment of $i$-sets in squashed order.

By repeated applications of Lemma 2.17, $(\triangle^{(i)} \mathcal{A})' = \nabla^{(n-i)} \mathcal{A}'$. Thus, by Observation 2.2, $(\triangle^{(i)} \mathcal{A} \cup \mathcal{A}^{(i)})' = (\triangle^{(i)} \mathcal{A})' \cup (\mathcal{A}^{(i)})' = \nabla^{(n-i)} \mathcal{A}' \cup \mathcal{A}^{(n-i)}$ is an initial segment of $(n-i)$-subsets of $[n]$ in antilexicographic order where the sets in $\nabla^{(n-i)} \mathcal{A}'$ follow the sets in $\mathcal{A}^{(n-i)}$ in squashed order. This holds for each $i$ such that $n-h \leq n-i \leq n-l$.

It follows that $\mathcal{A}'$ is an antilexicographic antichain on $[n]$.

Assume now that $\mathcal{A}$ is an antilexicographic antichain on $[n]$. The proof that $\mathcal{A}'$ is a squashed antichain on $[n]$ is done by using a similar argument to the one in the last paragraph. The lemma follows. 

$\blacksquare$

2.12 A Miscellany

Completely Separating Systems

Some results dealing with the relationship between antichains and completely separating systems are included here. This relationship is discussed in depth in [20]. A review and major results on completely separating systems can be found in [19, 25, 26]. We begin by a simple observation.

Observation 2.87. If $\mathcal{C}$ is a $(n)CSS$ and $\mathcal{A}$ is its dual then $V(\mathcal{C}) = V(\mathcal{A})$.

The next theorem describes the relationship between antichains and completely separating systems.
Theorem 2.88 (Spencer [28]). \( C \) is a completely separating system if and only if the dual \( A \) of \( C \) is an antichain.

It follows that

\textbf{Observation 2.89.} If \( C \) is a \((n)\)CSS and \( A \) is its dual then \( A \) is an antichain on \([|C|]\) of size \( n \).

\textbf{Observation 2.90.} If \( C \) is a \((n,k)\)CSS and \( A \) is its dual then \( A \) is an antichain on \([|C|]\) of size \( n \) where each \( i \in [|C|] \) occurs in exactly \( k \) members of \( A \).

\textbf{Example 2.91.} \( C = \{124, 135, 236, 456\} \) is a minimal \((6,3)\)CSS. The dual of \( C \) is \( A = \{12, 13, 23, 14, 24, 34\} \). It is easy to check that \( A \) is an antichain on \([4]\) of size \( 6 \) and that \( V(C) = 11 = V(A) \). In addition, each element in \([4]\) occurs in 3 members of \( A \).

\section*{Convex Functions}

Some results in this thesis are proved by making use of the property of convex functions. The following is a well-known result.

\textbf{Theorem 2.92 ([31, p. 2]).} Let \( f : D \rightarrow \mathbb{R} \) be a convex function. Let \( i \in D \), \( \lambda_i \in \mathbb{R}^+ \cup \{0\} \), and \( \sum_{i=0}^{n} \lambda_i = 1 \). Assume that \( \sum_{i=0}^{n} \lambda_i i \in D \). Then

\[ f \left( \sum_{i=0}^{n} \lambda_i i \right) \leq \sum_{i=0}^{n} \lambda_i f(i). \]
Chapter 3

On Shadows and Shades of Collections of Sets: Stronger Versions of Theorems 2.47 and 2.48
Chapter 3. Stronger Versions of Theorems 2.47 and 2.48

3.1 New Results: Theorems 3.1 and 3.2

In this chapter stronger versions of Theorems 2.47 and 2.48 are given as Theorems 3.1 and 3.2 respectively. The strengthening in each case consists of showing that the first inequality in each of Theorems 2.47 and 2.48 is a strict inequality. A corollary of Theorem 3.1, namely Corollary 3.5, is needed to prove a result in Chapter 8 (see Theorem 8.8).

Theorem 3.1. Let \( p \in \mathbb{Z}^+ \) be such that \( p \leq \binom{n}{k} \). Assume that \( C_{n,k}(p) \neq F_{n,k}(p) \). Then

\[
|\Delta F_{n,k}(p)| > |\Delta N C_{n,k}(p)|.
\]

Theorem 3.2. Let \( p \in \mathbb{Z}^+ \) be such that \( p \leq \min\{\binom{n}{k}, \binom{n}{k+1}\} \). Then

\[
|\Delta F_{n,k}(p)| < |\Delta F_{n,k+1}(p)|.
\]

Theorem 3.1 says that amongst all the collections \( B \) of \( p \) consecutive \( k \)-sets in squashed order, \( F_{n,k}(p) \) is the unique collection which maximises \( |\Delta N B| \) since \( |\Delta F_{n,k}(p)| = |\Delta N F_{n,k}(p)| \) by Observation 2.9. In addition, Theorem 3.2 says that, for \( k \in \mathbb{Z}^+ \), \( |\Delta N B| \) is maximum if and only if \( B = F_{n,k}(p) \) where \( k \) is the largest possible integer for which \( p \leq \binom{n}{k} \).

The proofs of Theorems 3.1 and 3.2 are given in Section 3.2. Theorems 2.47 and 2.48 have been proved by Clements [8] for multisets in lexicographic order. In Section 3.3 it is shown that Theorems 3.1 and 3.2 do not hold for multisets. For the remainder of this section we present some corollaries of Theorems 3.1 and 3.2. The first two corollaries are obtained by replacing \( k \) by \( n - k \) in Theorem 3.1 and \( k \) by \( n - k - 1 \) in Theorem 3.2 and applying Lemma 2.21(ii) and Observation 2.9.

Corollary 3.3. Let \( p \in \mathbb{Z}^+ \) be such that \( p \leq \binom{n}{k} \). Assume that \( C_{n,k}(p) \neq L_{n,k}(p) \). Then

\[
|\nabla L_{n,k}(p)| > |\nabla N C_{n,k}(p)|.
\]
Corollary 3.4. Let \( p \in \mathbb{Z}^+ \) be such that \( p \leq \min\{\binom{n}{k}, \binom{n}{k+1}\} \). Then

\[
\left| \nabla L_{n,k}(p) \right| > \left| \nabla L_{n,k+1}(p) \right|.
\]

The next corollary improves upon Corollary 2.53.

Corollary 3.5. Let \( p_1, p_2 \in \mathbb{Z}^+ \) be such that \( p_1 + p_2 \leq \binom{n}{k} \). Then

\[
|\Delta F_{n,k}(p_1 + p_2)| < |\Delta F_{n,k}(p_1)| + |\Delta F_{n,k}(p_2)|.
\]

Proof. Applying Theorem 3.1 instead of Theorem 2.47 in the proof of Corollary 2.53 gives the result. \( \square \)

3.2 The Proofs of Theorems 3.1 and 3.2

The proofs of Theorems 3.1 and 3.2 are adaptations of Clements’ proofs of Theorems 2.47 and 2.48. Both proofs use induction on \( n \). To prove Theorem 3.1 one assumes that Theorems 3.1 and 3.2 hold for \( n - 1 \). To prove Theorem 3.2 one assumes that Theorem 3.1 holds for \( n \) and that Theorem 3.2 holds for \( n - 1 \). We thus state the induction hypotheses as follows.

Induction Hypothesis 3.6 (Induction Hypothesis for Theorem 3.1).

(i) Theorem 3.1 holds for \( n - 1 \).

(ii) Theorem 3.2 holds for \( n - 1 \).

Induction Hypothesis 3.7 (Induction Hypothesis for Theorem 3.2).

(i) Theorem 3.1 holds for \( n \).

(ii) Theorem 3.2 holds for \( n - 1 \).

We prove Theorem 3.1 first. The ordering on sets in the proofs below is assumed to be the squashed order.

Proof of Theorem 3.1. It is easy to verify that Theorems 3.1 and 3.2 hold for \( n = 1 \) and \( n = 2 \). Assume that Induction Hypothesis 3.6 holds.
Consider a collection \( C_{n,k}(p) \) of \( p \) consecutive \( k \)-subsets of \([n]\) such that \( C_{n,k}(p) \neq F_{n,k}(p) \). Let \( C_{n,k}(p) = \mathcal{P}_1 \cup \mathcal{P}_2 \) where \( \mathcal{P}_1 = F_{n,k} \binom{n-1}{k} \cap C_{n,k}(p) \) and \( \mathcal{P}_2 = L_{n,k} \binom{n-1}{k-1} \cap C_{n,k}(p) \). That is, \( \mathcal{P}_1 \) consists of \(|\mathcal{P}_1|\) consecutive \( k \)-subsets of \([n-1]\) and \( \mathcal{P}_2 \) consists of \(|\mathcal{P}_2|\) consecutive \( k \)-subsets of \([n]\) each having \( n \) as an element. Then \( \mathcal{P}_1 = C_{n-1,k}(|\mathcal{P}_1|) \) and, by Lemma 2.30, \( \mathcal{P}_2 \) corresponds to a collection \( C_{n-1,k-1}(|\mathcal{P}_2|) \).

We consider two cases.

(i) Assume that \( \mathcal{P}_2 = \emptyset \). Then \( p < \binom{n-1}{k} \) and \( C_{n,k}(p) = \mathcal{P}_1 = C_{n-1,k}(p) \). Also, \( F_{n,k}(p) = F_{n-1,k}(p) \) by Observation 2.6 as \( p < \binom{n-1}{k} \). It follows that

\[
F_{n-1,k}(p) \neq C_{n-1,k}(p)
\]

since \( C_{n,k}(p) \neq F_{n,k}(p) \) by assumption.

Now,

\[
\left| \Delta F_{n,k}(p) \right| = \left| \Delta F_{n-1,k}(p) \right|
\]

(by Lemma 2.28 as \( p < \binom{n-1}{k} \))

\[
> \left| \Delta N C_{n-1,k}(p) \right|
\]

(by (3.1) and Induction Hypothesis 3.6 (i))

\[
= \left| \Delta N C_{n,k}(p) \right|
\]

(as \( C_{n,k}(p) = C_{n-1,k}(p) \))

(ii) Assume that \( \mathcal{P}_2 \neq \emptyset \). Define \( \mathcal{P}_3 \) to be

\[
\mathcal{P}_3 = \begin{cases} 
\mathcal{P}_2 & \text{if } |\mathcal{P}_2| \leq \binom{n-1}{k} - |\mathcal{P}_1|, \\
L \left( \binom{n-1}{k} - |\mathcal{P}_1|, |\mathcal{P}_2| \right) & \text{if } |\mathcal{P}_2| > \binom{n-1}{k} - |\mathcal{P}_1|.
\end{cases}
\]

In both cases \( \mathcal{P}_3 \subseteq \mathcal{P}_2 \), so that \( \mathcal{P}_3 \) corresponds to a collection \( C_{n-1,k-1}(|\mathcal{P}_3|) \). If \( \mathcal{P}_3 = \mathcal{P}_2 \) then \( |\mathcal{P}_3| > 0 \). If \( \mathcal{P}_3 = L \left( \binom{n-1}{k} - |\mathcal{P}_1|, |\mathcal{P}_2| \right) \) then \( |\mathcal{P}_3| > 0 \) as well as \( |\mathcal{P}_1| < \binom{n-1}{k} \) since \( C_{n,k}(p) \neq F_{n,k}(p) \). Note also that \( |\mathcal{P}_3| \leq \binom{n-1}{k} \) by the definition of \( \mathcal{P}_3 \).
We first show that $|\Delta F_{n,k}([P_3])| > |\Delta N P_3|:

$$
|\Delta F_{n,k}([P_3])| = |\Delta F_{n-1,k}([P_3])|
$$

(by Lemma 2.28 as $|P_3| \leq \binom{n-1}{k}$)

$$
> |\Delta F_{n-1,k-1}([P_3])|
$$

(by Induction Hypothesis 3.6 (ii)

as $|P_3| > 0$)

$$
\geq |\Delta N C_{n-1,k-1}([P_3])|
$$

(by Theorem 2.47)

$$
= |\Delta N P_3|.
$$

(as $P_3$ corresponds to $C_{n-1,k-1}([P_3])$)

To complete the proof of Case (ii) three cases must be considered.

(a) Assume that $|P_2| > \binom{n-1}{k} - |P_1|$ and $P_1 \neq \emptyset$. Then $P_3 = L\left(\binom{n-1}{k} - |P_1|, P_2\right)$. Replace $P_3$ by $F_{n,k}([P_3])$ to obtain $F_{n,k}(p)$ as $F_{n,k}(p)$ is the union of the disjoint collections $F_{n,k}([P_3])$, $P_1$ and $P_2 \setminus P_3$. Then

$$
|\Delta F_{n,k}(p)| = |\Delta F_{n,k}([P_3])| + |\Delta N P_1| + |\Delta N (P_2 \setminus P_3)|
$$

(by construction and Observation 2.11)

$$
> |\Delta N P_3| + |\Delta N P_1| + |\Delta N (P_2 \setminus P_3)|
$$

(by (3.2))

$$
= |\Delta N C_{n,k}(p)|.
$$

(by Observation 2.10)

(b) Assume that $|P_2| > \binom{n-1}{k} - |P_1|$ and $P_1 = \emptyset$. Then $P_3 = L\left(\binom{n-1}{k}, P_2\right)$. Recall that $P_2$ corresponds to a collection $C_{n-1,k-1}([P_2])$. It follows that $P_2 \setminus P_3$ corresponds to a collection $C_{n-1,k-1}([P_2 \setminus P_3])$. Replace $P_3$ by $F_{n,k}([P_3]) = F_{n,k}\left(\binom{n-1}{k}\right)$ and $P_2 \setminus P_3$
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by \( F_{n-1,k-1}([P_2 \setminus P_3]) \uplus \{a\} \) to obtain \( F_{n,k}(p) \). Then

\[
|\Delta F_{n,k}(p)| = |\Delta F_{n,k}([P_3])| + |\Delta N_{n-1,k-1}([P_2 \setminus P_3])|
\]

(by construction, Observation 2.11 and Lemma 2.30)

\[
> |\Delta N\mathcal{P}_3| + |\Delta N_{n-1,k-1}([P_2 \setminus P_3])|
\]

(by (3.2))

\[
\geq |\Delta N\mathcal{P}_3| + |\Delta N C_{n-1,k-1}([P_2 \setminus P_3])|
\]

(by Observation 2.9 and Theorem 2.47)

\[
= |\Delta N\mathcal{P}_3| + |\Delta N([P_2 \setminus P_3])|
\]

(by Lemma 2.30)

as \( P_2 \setminus P_3 \) corresponds to \( C_{n-1,k-1}([P_2 \setminus P_3]) \).

\[
= |\Delta N C_{n,k}(p)|.
\]

(by Observation 2.10)

(c) Assume that \( |P_2| \leq \binom{n-1}{k} - |P_1| \). Then \( P_3 = P_2 \). Recall that \( P_1 = C_{n-1,k}([P_1]) \). Moreover, as \( P_2 \neq \emptyset \), \( P_1 = L_{n-1,k}([P_1]) \). Replace \( P_3 \) by \( F_{n,k}([P_3]) \) and \( P_1 \) by \( N_{n,k}^{[P_3]}([P_1]) \) to obtain \( F_{n,k}(p) \). Note that, by Observation 2.6, \( F_{n,k}([P_3]) \) and \( N_{n,k}^{[P_1]}([P_1]) \) are \( F_{n-1,k}([P_3]) \) and \( N_{n-1,k}^{[P_1]}([P_1]) \) respectively as \( |P_3| + |P_1| \leq \binom{n-1}{k} \). Then

\[
|\Delta F_{n,k}(p)| = |\Delta F_{n,k}([P_3])| + |\Delta N N_{n,k}^{[P_1]}([P_1])|
\]

(by construction and Observation 2.11)

\[
> |\Delta N\mathcal{P}_3| + |\Delta N N_{n,k}^{[P_1]}([P_1])|
\]

(by (3.2))

\[
= |\Delta N\mathcal{P}_3| + |\Delta N N_{n-1,k}^{[P_1]}([P_1])|
\]

(as \( N_{n,k}^{[P_1]}([P_1]) = N_{n-1,k}^{[P_1]}([P_1]) \))

\[
\geq |\Delta N\mathcal{P}_3| + |\Delta N L_{n-1,k}([P_1])|
\]

(by Note 2.52 and Theorem 2.47)
That is,

\[
|\Delta F_{n,k}(p)| \geq |\Delta_N P_3| + |\Delta_N L_{n-1,k}([P_1])|
\]

\[
= |\Delta_N P_3| + |\Delta_N P_1|
\]

(as \( P_1 = L_{n-1,k}([P_1]) \))

\[
= |\Delta_N C_{n,k}(p)|.
\]

(by Observation 2.10)

This concludes the proof of Theorem 3.1.

\[ \square \]

**Proof of Theorem 3.2.** It is easy to verify that Theorem 3.1 holds for \( n = 1, 2, 3 \) and that Theorem 3.2 holds for \( n = 1, 2 \). Assume that Induction Hypothesis 3.7 holds.

Let \( p \) be such that \( 0 < p \leq \min\{(\tbinom{n}{k}), (\tbinom{n}{k+1})\} \). We consider two cases.

(i) Assume that \( p \leq (\tbinom{n-1}{k}) \). Let \( P_1 \) be the collection of \( p \) consecutive \( k \)-sets that come immediately after the sets in \( F_{n,k+1}(\tbinom{n-1}{k+1}) \). Then \( P_1 \neq F_{n,k+1}(p) \) and \( P_1 \) is a collection \( C_{n,k+1}(p) \). In addition, the sets in \( P_1 \) are the first \( (k+1) \)-subsets of \([n]\) each having \( n \) as an element. Thus \( P_1 \) corresponds to \( F_{n-1,k}(p) \) by Lemma 2.30. Then

\[
|\Delta F_{n,k+1}(p)| > |\Delta_N P_1|
\]

(by Induction Hypothesis 3.7 (i))

as \( P_1 = C_{n,k+1}(p) \) and \( P_1 \neq F_{n,k+1}(p) \)

\[
= |\Delta_N F_{n-1,k}(p)|
\]

(as \( P_1 \) and \( F_{n-1,k}(p) \) correspond)

\[
= |\Delta F_{n,k}(p)|.
\]

(by Lemma 2.28 and Observation 2.9)

(ii) Assume that \( p > (\tbinom{n-1}{k}) \). Partition \( F_{n,k}(p) \) into \( P_2 \) and \( P_3 \) such that \( P_2 = F_{n,k}(\tbinom{n-1}{k}) \) and \( P_3 = F_{n,k}(p) \setminus P_2 \). Then \( P_2 \) corresponds to \( F_{n-1,k}(\tbinom{n-1}{k}) \) by Lemma 2.28.

Therefore, \( P_2 \) corresponds to \( L_{n-1,k}(\tbinom{n-1}{k}) \) by Observation 2.7. It follows that

\[
P_2 \text{ corresponds to } L_{n,k+1}([P_2]) \tag{3.3}
\]
by Lemma 2.31. Note that $\mathcal{P}_3 \neq \emptyset$ and that $\mathcal{P}_3$ consists of the first $|\mathcal{P}_3|$ $k$-subsets of $[n]$ each having $n$ as an element. Therefore

$$\mathcal{P}_3 \text{ corresponds to } F_{n-1,k-1}(|\mathcal{P}_3|)$$

by Lemma 2.30. Note that $|\mathcal{P}_3| \leq \binom{n-1}{k+1}$ as $p \leq \min\{\binom{n}{k}, \binom{n}{k+1}\}$ by assumption. Now,

$$|\Delta F_{n,k+1}(p)| = |\Delta F_{n,k+1}(|\mathcal{P}_3|) + |\mathcal{P}_2|)|$$

$$= |\Delta F_{n,k+1}(|\mathcal{P}_3|) + |\Delta_N N_{n,k+1}^{\mathcal{P}_3}(|\mathcal{P}_2|)|$$

(by Observation 2.11)

$$= |\Delta F_{n-1,k+1}(|\mathcal{P}_3|) + |\Delta_N N_{n,k+1}^{\mathcal{P}_3}(|\mathcal{P}_2|)|$$

(by Lemma 2.28 as $|\mathcal{P}_3| \leq \binom{n-1}{k+1}$)

$$> |\Delta F_{n-1,k-1}(|\mathcal{P}_3|) + |\Delta_N N_{n,k+1}^{\mathcal{P}_3}(|\mathcal{P}_2|)|$$

(by Induction Hypothesis 3.7.(ii) as $|\mathcal{P}_3| > 0$)

$$\geq |\Delta F_{n-1,k-1}(|\mathcal{P}_3|) + |\Delta N L_{n,k+1}(|\mathcal{P}_2|)|$$

(by Note 2.52 and Theorem 2.47)

$$= |\Delta_N \mathcal{P}_2| + |\Delta_N \mathcal{P}_3|$$

(by (3.3) and (3.4))

$$= |\Delta F_{n,k}(p)|.$$  

(by construction and Observation 2.10)

This concludes the proof of Theorem 3.2. 

\[ \square \]

### 3.3 Multisets and Theorems 3.1 and 3.2

As mentioned in Section 3.1 Theorems 2.47 and 2.48 were proved for multisets in lexicographic order. We show in the next two examples that the strengthened forms of Theorems 2.47 and 2.48, namely Theorems 3.1 and 3.2 respectively, do not hold for multisets in lexicographic order.
Example 3.8. Consider the multiset \( S(2, 3, 4) \). Referring to Example 1.14, the collection of the first two vectors of rank 3 in lexicographic order is \( \{(0, 0, 3), (0, 1, 2)\} \). We choose \( \{(1, 0, 2), (1, 1, 1)\} \) to be the collection of two consecutive 3-vectors in lexicographic order whose new-shadow will be compared to that of \( \{(0, 0, 3), (0, 1, 2)\} \).

As for sets, the new-shadow \( \triangle_N m \) of a vector \( m \) is defined to be the collection of the vectors in \( \triangle m \) which are not in the shadow of any vector which precedes \( m \) in lexicographic order. For each vector, we list the vectors in its new-shadow:

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \triangle_N m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 3</td>
<td>0 0 2</td>
</tr>
<tr>
<td>0 1 2</td>
<td>0 1 1</td>
</tr>
<tr>
<td>1 0 2</td>
<td>1 0 1</td>
</tr>
<tr>
<td>1 1 1</td>
<td>1 1 0</td>
</tr>
</tbody>
</table>

It follows that \( |\triangle_N \{(0, 0, 3), (0, 1, 2)\}| = |\triangle_N \{(1, 0, 2), (1, 1, 1)\}| \) and that Theorem 3.1 does not hold in this case.

Example 3.9. Consider the multiset \( S(2, 3, 4) \). The first 3-vector in lexicographic order is \((0, 0, 3)\) and its shadow is \( \{(0, 0, 2)\} \). The first 4-vector in lexicographic order is \((0, 0, 4)\) and its shadow is \( \{(0, 0, 3)\} \). This shows that Theorem 3.2 does not hold in this case.
Chapter 4

The Flat Antichain Conjecture
4.1 The Conjecture

In this chapter we turn our attention to the problem which is the main motivation for this thesis. While investigating the size of a minimal completely separating system, the following problem arose

Given $n$, $s$ and $V \in \mathbb{Z}^+$, does there exist an antichain on $[n]$ of size $s$ and volume $V$?

In an attempt to answer this question, it was conjectured in [19] that there exists an antichain $A$ on $[n]$ if and only if there exists a flat antichain on $[n]$ of the same size and volume as $A$. That is

**Conjecture 4.1 (The flat antichain conjecture, Lieby [19]).** For any antichain $A$ on $[n]$ there exists a flat antichain $A^*$ on $[n]$ such that $|A^*| = |A|$ and $V(A^*) = V(A)$.

The conjecture has an alternative statement as

**Alternative statement of the flat antichain conjecture:** $A$ is an antichain on $[n]$ if and only if there exists a flat counterpart of $A$ which is an antichain on $[n]$.

The conjecture has been proven to hold in some special cases. In [23] Maire proved that the flat antichain conjecture holds for antichains whose average set size is an integer.

**Theorem 4.2 (Maire [23]).** Let $A$ be an antichain on $[n]$ with $\bar{A} = k$ for some integer $k$. Then the flat antichain conjecture holds for $A$.

We will see in Section 5.2 that Theorem 4.2 can also be derived from a result by Kleitman [17]. Another special case of the conjecture has been solved by Roberts [27].

**Theorem 4.3 (Roberts [27]).** The flat antichain conjecture holds for antichains $A$ on $[n]$ with $\bar{A} \leq 3$. 
In this thesis Theorems 6.7, 8.2, 8.7, 8.14, and 8.19 and Corollaries 8.4 and 8.5 prove that the flat antichain conjecture is true in seven further cases. In Section 4.2 below we present the original motivation for the flat antichain conjecture. In Section 4.3 it is shown that if \(A\) is an antichain then the parameters of any flat counterpart of \(A\) satisfy the LYM inequality. The following simple observation is a direct consequence of Observation 2.84.

Observation 4.4. Let \(A\) be an antichain on \([n]\). If the flat antichain conjecture holds for \(A\) then it also holds for \(A'\), the complement of \(A\).

4.2 A Motivation for the Flat Antichain Conjecture

Given \(n\) and \(k\), a generic problem is to determine the size \(R(n,k)\) of a minimal \((n,k)\)CSS. Ramsey et al. in [25, 26] give bounds for \(R(n,k)\) for all \(n\) and \(k\). Exact values for \(R(n,k)\) are known in some cases. The flat antichain conjecture arose from attempts to determine \(R(n,k)\) for some given \(n\) and \(k\) as explained below.

Let \(n\) and \(k\) be given and assume that \(h\) and \(l\) are upper and lower bounds on \(R(n,k)\) respectively. These bounds are known and are found in [25, 26]. The problem is to determine \(R(n,k)\). In particular, the problem is to decide if \(R(n,k) = R\), where \(R\) is some appropriately chosen value in the interval \([h,l]\). This is equivalent to determining if a minimal \((n,k)\)CSS of size \(R\) exists, or, to simplify the problem, if a \((n,k)\)CSS of size \(R\) exists.

Let \(R \in [h,l]\). By Observation 2.90, determining if there exists a \((n,k)\)CSS \(C\) with \(|C| = R\) is equivalent to determining if there exists an antichain on \([R]\) of size \(n\) in which each element in \([R]\) occurs exactly \(k\) times. The problem of deciding upon the existence of antichains where each element occurs the same number of times appears to be very difficult. A simpler problem is to be able to determine if an antichain of a specified size and volume exists. For, if \(|C| = R\), then the dual of \(C\) is an antichain on \([R]\) of volume \(V(C) = kR\) by Observation 2.87. Thus a necessary condition for \(|C| = R\) is that there exists an antichain on \([R]\) of size \(n\) and volume \(kR\).
Chapter 4. The Flat Antichain Conjecture

Theorem 2.77 states necessary and sufficient conditions for some integers to be the parameters of an antichain on \([n]\). In the present case, the problem is to determine whether the integers \(n\) and \(kR\) are the size and volume of an antichain on \([R]\). Theorem 2.77 can be used to solve this problem by using Algorithm 4.5 below. Let \(n\), \(s\) and \(V\) be given. Algorithm 4.5 tests if \(s\) and \(V\) are the size and volume of an antichain on \([n]\).

Algorithm 4.5.

Step 1. Partition \(V\) into \(n\) parts \(P_i\) where \(P_i = i\ell, \ell > 0\), such that \(\sum_{i=0}^{n} p_i = s\).

Step 2. Check if the numbers \(p_i\) are the parameters of an antichain on \([n]\) by applying Theorem 2.77.

Step 3. If Theorem 2.77 is satisfied then there exists an antichain on \([n]\) with size \(s\) and volume \(V\).

If Theorem 2.77 is not satisfied then

- If all partitions of \(V\) as described in Step 1 have been found then stop,
- Else repeat Step 1.

Example 4.6. Let \(n = 5\), \(s = 8\), and \(V = 18\). The possible partitions of \(V\) with \(\sum_{i=0}^{n} i\ell; = V\) and \(\sum_{i=0}^{n} p_i = s\) are (the numbers in boldface are the \(p_i; s\)):

\[
\begin{align*}
V &= 5 \times 1 + 1 \times 3 + 2 \times 5 & V &= 3 \times 1 + 5 \times 3 \\
V &= 5 \times 1 + 2 \times 4 + 1 \times 5 & V &= 2 \times 1 + 4 \times 2 + 1 \times 3 + 1 \times 5 \\
V &= 4 \times 1 + 2 \times 2 + 2 \times 5 & V &= 2 \times 1 + 4 \times 2 + 2 \times 4 \\
V &= 4 \times 1 + 1 \times 2 + 1 \times 3 + 1 \times 4 + 1 \times 5 & V &= 2 \times 1 + 3 \times 2 + 2 \times 3 + 1 \times 4 \\
V &= 4 \times 1 + 1 \times 2 + 3 \times 4 & V &= 2 \times 1 + 2 \times 2 + 4 \times 3 \\
V &= 4 \times 1 + 3 \times 3 + 1 \times 5 & V &= 1 \times 1 + 6 \times 2 + 1 \times 5 \\
V &= 4 \times 1 + 2 \times 3 + 2 \times 4 & V &= 1 \times 1 + 5 \times 2 + 1 \times 3 + 1 \times 4 \\
V &= 3 \times 1 + 3 \times 2 + 1 \times 4 + 1 \times 5 & V &= 1 \times 1 + 4 \times 2 + 3 \times 3 \\
V &= 3 \times 1 + 2 \times 2 + 2 \times 3 + 1 \times 5 & V &= 7 \times 2 + 1 \times 4 \\
V &= 3 \times 1 + 2 \times 2 + 1 \times 3 + 2 \times 4 & V &= 6 \times 2 + 2 \times 3 \\
V &= 3 \times 1 + 1 \times 2 + 3 \times 3 + 1 \times 4 & V &= 6 \times 2 + 2 \times 3 
\end{align*}
\]

It can be easily checked that for each of the above partitions of \(V\) the numbers \(p_i\) do not satisfy the conditions of Theorem 2.77. Therefore there exists no antichain on \([5]\) with size 8 and volume 18.
Although Algorithm 4.5 will always enable us to decide whether $s$ and $V$ are the size and volume of an antichain on $[n]$, it is computationally expensive (see [24, p. 28] for a discussion of this issue). If the flat antichain conjecture holds, then an antichain on $[n]$ of size $s$ and volume $V$ exists if and only if a flat antichain on $[n]$ of the same size and volume exists. This is the basis for a more efficient algorithm.

Let $k$ and $r$ be the unique nonnegative integers such that $V = ks + r$, $0 \leq r < s$. The parameters $q_i$ of a flat collection of sets of size $s$ and volume $V$ are uniquely determined and are $q_i = 0$ for $i \neq k$, $k + 1$, $q_{k+1} = r$ and $q_k = s - r$. Thus, assuming that the flat antichain conjecture holds, determining if $s$ and $V$ are the size and volume of an antichain on $[n]$ is equivalent to determining if the $q_i$s are the parameters of an antichain on $[n]$. This is made formal in the following algorithm.

**Algorithm 4.7.**

Step 1. Let $k$ and $r$ be such that $V = ks + r$, $0 \leq r < s$.

Let $q_{k+1} = r$ and $q_k = s - r$.

Step 2. Check if $q_{k+1}$ and $q_k$ are the parameters of an antichain on $[n]$ by applying Theorem 2.77.

Step 3. Assume that the flat antichain conjecture holds.

Then there exists an antichain on $[n]$ with size $s$ and volume $V$ if and only if Theorem 2.77 is satisfied.

**Example 4.8.** In Example 4.6 $V = 2 \times 8 + 2$. There exists no flat antichain on $[5]$ with 2 3-sets and 6 2-sets. Thus, if the flat antichain conjecture holds, then there exists no antichain on $[5]$ with size 8 and volume 18.

Algorithm 4.7 is a polynomial time algorithm and is therefore more efficient than Algorithm 4.5. The reader is referred to Chapter 10 where, assuming that the flat antichain conjecture holds, necessary and sufficient conditions for the existence of antichains on $[n]$ with a given size and volume are stated (see Theorem 10.1).
4.3 The Parameters of any Flat Counterpart of an Antichain Satisfy LYM

Let \( \mathcal{A} \) be any antichain on \([n]\) and let \( \mathcal{A}^* \) be a flat counterpart of \( \mathcal{A} \) with parameters \( q_i \). Then the flat antichain conjecture holds if and only if the \( q_i \)s are the parameters of an antichain on \([n]\). This could be shown by applying Theorem 2.77. So far it has not yet been proved if the parameters of a flat counterpart of any antichain on \([n]\) satisfy Theorem 2.77 in the universal set \([n]\). However it can be proved that these parameters always satisfy the LYM inequality in the universal set \([n]\) as shown by Theorem 4.9 below. Recall that the LYM inequality states a necessary condition for the \( q_i \)s to be the parameters of an antichain on \([n]\). It is therefore not difficult to see that Theorem 4.9 verifies a necessary condition for the flat antichain conjecture to hold.

Theorem 4.9 and its proof are due to Maire [23] and Woods [32]. Theorem 4.9 uses a convexity argument.

**Theorem 4.9 (Maire [23], Woods [32]).** Let \( \mathcal{A} \) be an antichain on \([n]\) and let \( \mathcal{A}^* \) be a flat counterpart of \( \mathcal{A}^* \). Then the parameters of \( \mathcal{A}^* \) satisfy the LYM inequality.

**Proof.** The definition of the convex function \( f(x) \) on the real numbers and the proof of (4.4) below can also be found in [1, p. 62].

Let \( i \in [n] \) and define \( f(i) = \frac{1}{\binom{n}{i}} \). In [23] it is shown that \( f(i) \) is a convex function on \([n]\). We can extend \( f \) to a convex function defined on the real interval \([0,n]\) by linear interpolation. This is done as follows. Let \( x \in [0,n] \), \( i \in [n] \) and \( \lambda \in [0,1] \) be such that \( x = \lambda i + (1 - \lambda)(i + 1) \). Then define

\[
f(x) = \lambda f(i) + (1 - \lambda)f(i + 1)
\]

so \( f(x) \) is a convex function. That is, if \( x = \sum_{i=1}^{n} \lambda_i i \) where \( x \in [0,n] \), \( \sum_{i=0}^{n} \lambda_i = 1 \) and \( \lambda_i \in [0,1] \) then

\[
f(x) \leq \sum_{i=0}^{n} \lambda_i f(i)
\]
Chapter 4. The Flat Antichain Conjecture

by Theorem 2.92.

Let $\mathcal{A}$ be an antichain on $[n]$ with parameters $p_i$. Then
\[ \sum_{i=0}^{n} p_i f(i) \leq 1 \]  
by the LYM inequality. Let $|\mathcal{A}| = s$ and let $\mathcal{A}^*$ be a flat counterpart of $\mathcal{A}$ with parameters $q_i$. Let $k$ and $r$ be the unique integers such that $V(\mathcal{A}) = ks + r$, $0 \leq r < s$. Then $q_{k+1} = r$ and $q_k = s - r$. Since $\sum_{i=0}^{n} p_i f(i) = V(\mathcal{A}) = V(\mathcal{A}^*) = q_{k+1}(k+1) + q_k k$, it follows that $\overline{\mathcal{A}} = \sum_{i=0}^{n} p_i i = \frac{q_{k+1}}{s}(k+1) + \frac{q_k}{s}k$. Thus,
\[ f(\overline{\mathcal{A}}) = \frac{q_{k+1}}{s} f(k+1) + \frac{q_k}{s} f(k) \]  
(by 4.1) as $\overline{\mathcal{A}} = \frac{q_{k+1}}{s}(k+1) + \frac{q_k}{s}k$

\[ = f \left( \sum_{i=0}^{n} \frac{p_i}{s} i \right) \]  
(as $\overline{\mathcal{A}} = \sum_{i=0}^{n} \frac{p_i}{s} i$)

\[ \leq \frac{1}{s} \sum_{i=0}^{n} p_i f(i) \]  
(by 4.2)

\[ \leq \frac{1}{s}. \]  
(by 4.3)

That is
\[ f(\overline{\mathcal{A}}) \leq \frac{1}{|\mathcal{A}|} \]  
\[ (4.4) \]

and
\[ \frac{q_k}{(n)} \left( \frac{n}{k} \right) + \frac{q_{k+1}}{(n+1)} \left( \frac{n+1}{k+1} \right) \leq 1. \]  
\[ (4.5) \]

This proves the theorem.

As an immediate corollary of Theorem 4.9 we have

Corollary 4.10. Let $\mathcal{A}$ be an antichain on $[n]$ with $V(\mathcal{A}) = k|\mathcal{A}| + r$, $k \in \mathbb{Z}^+$, $0 \leq r < |\mathcal{A}|$. Then
\[ \frac{|\mathcal{A}| - r}{\left( \begin{array}{c} n \\rule{0pt}{7.5pt} \\ k \end{array} \right)} + \frac{r}{\left( \begin{array}{c} n \\rule{0pt}{7.5pt} \\ k+1 \end{array} \right)} \leq 1. \]
Chapter 4. The Flat Antichain Conjecture

Proof. This follows from (4.5) by noting that the parameters $q_i$ of a flat counterpart of $\mathcal{A}$ are $q_i = 0$ for $i \neq k + 1, k$, and $q_{k+1} = r, q_k = |\mathcal{A}| - r$. □

The following theorem relates the parameters of an antichain $\mathcal{A}$ to the parameters of a flat counterpart of $\mathcal{A}$.

Theorem 4.11. Let $\mathcal{A}$ be an antichain on $[n]$ with parameters $p_i$. Let $\mathcal{A}^*$ be a flat counterpart of $\mathcal{A}$ with parameters $q_i$. Then

$$\sum_{i=0}^{n} \frac{q_i}{\binom{n}{i}} \leq \sum_{i=0}^{n} \frac{p_i}{\binom{n}{i}}$$

Proof. This follows from the proof of Theorem 4.9 where it is shown that $f(\mathcal{A}) = \frac{q_{k+1}}{\binom{n}{k+1}} f(k + 1) + \frac{q_k}{\binom{n}{k}} f(k) \leq \frac{1}{|\mathcal{A}|} \sum_{i=0}^{n} p_i f(i)$. That is, $\frac{q_{k+1}}{\binom{n}{k+1}} + \frac{q_k}{\binom{n}{k}} \leq \sum_{i=0}^{n} \frac{p_i}{\binom{n}{i}}$ as required. □

Theorem 4.11 is a reassuring result in the sense that it shows that the LYM inequality is no tighter for the parameters of a flat counterpart of an antichain $\mathcal{A}$ than it is for the parameters of $\mathcal{A}$. However, the LYM inequality is only a necessary condition for some integers to be the parameters of an antichain, so Theorems 4.9 and 4.11 are not sufficient to prove that the flat antichain conjecture holds.

The convexity argument used in the proof of Theorem 4.9 is a very powerful argument. From (4.4) two additional results can be derived. One is Theorem 4.2 which Maire [23] proved by using (4.4) as follows. Assume that $\mathcal{A}$ is an antichain on $[n]$ with $\mathcal{A} = k$, $k \in \mathbb{Z}^+$. Then replacing $\mathcal{A}$ by $k$ in (4.4) gives the inequality $\frac{1}{\binom{n}{k}} \leq \frac{1}{|\mathcal{A}|}$. This ensures that a flat antichain with $|\mathcal{A}| \mathcal{A}$-sets exists, thus proving Theorem 4.2. The second result is Theorem 5.1 by Kleitman and Milner [17]. Theorem 5.1 is stated in the next chapter where it is shown how it can be obtained from a variant of (4.4).
Chapter 5

Volumes of Antichains
Chapter 5. Volumes of Antichains

5.1 Introduction

This chapter investigates antichains from the viewpoint of their volumes. There are few results in the literature which concern the volume of an antichain. Two of those results are presented here. The first result, Theorem 5.1, is by Kleitman and Milner [17] and is presented in Section 5.2. Theorem 5.1 gives a lower bound for the average set size of an antichain. We give an alternative proof of Theorem 5.1 which uses the convexity argument already encountered while proving Theorem 4.9. An immediate consequence of Theorem 5.1 is that there is a profile-unique flat antichain of size \( \binom{n}{k} \) and volume \( k \binom{n}{k} \). In addition, Theorem 5.1 can be used to give an alternative proof of Theorem 4.2 which shows that the flat antichain conjecture holds for antichains whose average set size is an integer.

The second result, Theorem 5.4, is by Clements [9] and is stated and discussed in Section 5.3. Theorem 5.4 shows that among the antichains in \( \Lambda_{n,s} \) which simultaneously achieve minimum volume and minimum size ideal in \( \Lambda_{n,s} \), there exists a flat antichain. The arguments Clements used in the proof of Theorem 5.4 enable us to show that any antichain in \( \Lambda_{n,s} \) which achieves minimum volume in \( \Lambda_{n,s} \) is flat (see Theorem 5.8). That is, there is a profile-unique flat antichain in \( \Lambda_{n,s} \) which achieves minimum volume in \( \Lambda_{n,s} \). It follows that an antichain in \( \Lambda_{n,s} \) which achieves maximum volume in \( \Lambda_{n,s} \) is also profile-unique and flat (see Corollary 5.9). Theorem 5.8 can be used to show that Theorem 5.1 is a special case of Theorem 5.4. Section 5.3 concludes by investigating the relationship between the antichains in \( \Lambda_{n,s} \) which achieve \( V_{\min}(\Lambda_{n,s}) \) and those which achieve \( I_{\min}(\Lambda_{n,s}) \). In particular, it is shown that there is no profile-unique antichain in \( \Lambda_{n,s} \) which achieves minimum size ideal in \( \Lambda_{n,s} \).

Theorem 5.8 is important in the sense that it shows that the volume \( V_{\min}(\Lambda_{n,s}) \) can only be achieved by a flat antichain. This suggests that some other volumes of antichains in \( \Lambda_{n,s} \) may only be achieved by antichains in \( \Lambda_{n,s} \) which are flat. We state this as a conjecture in Section 5.4.
5.2 A Profile-Unique Antichain in $\Lambda_{n, \binom{n}{k}}$ with Volume $k\binom{n}{k}$

In [16] Kleitman shows that the average set size of an antichain $A$ on $[n]$ is at least $k$ if $|A| \geq \binom{n}{k}$ and $k \leq \frac{n}{2}$. In [17] Kleitman and Milner improve this result as follows.

**Theorem 5.1 (Kleitman and Milner [17]).** Let $A$ be an antichain on $[n]$, with $|A| \geq \binom{n}{k}$ and $k \leq \frac{n}{2}$. Then

(i) $\overline{A} \geq k$,

(ii) equality holds in (i) if and only if $A$ consists of all the $\binom{n}{k}$ $k$-sets.

The proof of Theorem 5.1.(i) in [16] and [17] uses Sperner’s lemma. A simpler proof, which can also be found in [1, p. 62], has been suggested by Spencer [29] and Woods [32]. The alternative proof of Theorem 5.1.(i) is essentially the same as the proof of Theorem 4.9 which shows that the parameters of a flat counterpart of an antichain satisfy the LYM inequality. The proof of Theorem 4.9 has been adapted to show that Theorem 5.1.(ii) holds as well.

**Alternative proof for Theorem 5.1.** Let $i \in [n]$ and $x$ and $\lambda$ be reals with $x \in [0, n]$ and $\lambda \in [0, 1]$. The function $f(x)$ is defined as in the proof of Theorem 4.9. That is, $f(i) = \frac{1}{\binom{n}{i}}$ and $f(x) = \lambda f(i) + (1 - \lambda)f(i + 1)$ for $x = \lambda i + (1 - \lambda)(i + 1)$. Then $f(x)$ is a convex function as stated in the proof of Theorem 4.9, and $f(x)$ is a strictly decreasing function on the interval $[0, \frac{n}{2}]$.

Let $A$ be an antichain on $[n]$ with parameters $p_i$. Assume that $|A| \geq \binom{n}{k}$ for some $k \leq \frac{n}{2}$. Two cases are considered.

(i) Assume that $A \neq [n]^l$ for $0 \leq l \leq n$. Then

$$\sum_{i=0}^{n} p_i f(i) < 1 \quad (5.1)$$
by Theorem 2.76. It follows that
\[
    f(\mathcal{A}) = f \left( \sum_{i=0}^{n} \frac{p_i}{|\mathcal{A}|^i} \right)
\]
\[
    \leq \frac{1}{|\mathcal{A}|} \sum_{i=0}^{n} p_i f(i)
\]
(by Theorem 2.92)
\[
    < \frac{1}{|\mathcal{A}|}.
\]
(by (5.1))

Thus \( f(\mathcal{A}) < \frac{1}{\binom{n}{k}} = f(k) \) and \( \mathcal{A} > k \) as \( f(x) \) is strictly decreasing on \([0, \frac{n}{2}]\).

(ii) Assume that \( \mathcal{A} = [n]^l \) for some \( l \) in the range \( k \leq l \leq n - k \). As \( |\mathcal{A}| \geq \binom{n}{k} \) and \( k \leq \frac{n}{2} \), those values of \( l \) are the only values for which it is possible that \( \mathcal{A} = [n]^l \). Then \( \mathcal{A} = k \) for \( l = k \) and \( \mathcal{A} > k \) for \( l > k \). This completes the proof of Theorem 5.1. \( \square \)

A simple corollary of Theorem 5.1 is the following.

**Corollary 5.2.** Let \( k \leq \frac{n}{2} \). A profile-unique antichain \( \mathcal{A} \) in \( \Lambda_{n,\binom{n}{k}} \) with volume \( k \binom{n}{k} \) is \( \mathcal{A} = [n]^k \). Moreover, \( V(\mathcal{A}) = V_{\min}(\Lambda_{n,\binom{n}{k}}) \).

**Proof.** Let \( \mathcal{A} \in \Lambda_{n,\binom{n}{k}} \). If \( \mathcal{A} \neq [n]^k \) then \( \overline{\mathcal{A}} > k \) by Theorem 5.1.(ii) and \( V(\mathcal{A}) > k \binom{n}{k} \). If \( \mathcal{A} = [n]^k \) then \( V(\mathcal{A}) = k \binom{n}{k} \). \( \square \)

**Note 5.3.**

The profile-unique antichain \( \mathcal{A} \) in Corollary 5.2 is actually the unique antichain in \( \Lambda_{n,\binom{n}{k}} \) with volume \( k \binom{n}{k} \).

This section concludes with an alternative proof of Theorem 4.2 which uses Theorem 5.1. Maire’s proof of Theorem 4.2 in [23] uses the LYM inequality and the convexity argument used to prove Theorems 4.9 and 5.1. Katona [15] suggested that Theorem 4.2 can also be derived from Theorem 5.1. This is shown below.

**Alternative Proof for Theorem 4.2.** Let \( \mathcal{A} \) be an antichain on \([n]\) with \( \overline{\mathcal{A}} = k \in \mathbb{Z}^+ \).

Assume that \( k > \frac{n}{2} \). By Observation 2.82, the complement \( \mathcal{A}' \) of \( \mathcal{A} \) is an antichain
of size $|A|$ with $\overline{\mathcal{F}} = n - k < \frac{n}{2}$. By Observation 4.4, if the flat antichain conjecture holds for $A'$ then it holds for $A$. Thus we can assume without loss of generality that $k \leq \frac{n}{2}$.

If $|A| > \binom{n}{k}$ then $\overline{\mathcal{A}} > k$ by Theorem 5.1.(ii). This contradicts the assumption about $\mathcal{A}$. Hence $|A| \leq \binom{n}{k}$ and a flat antichain on $[n]$ with $|A|$ $k$-sets exists.

\[\Box\]

### 5.3 A Profile-Unique Antichain in $\Lambda_{n,s}$ with Volume $V_{\min}(\Lambda_{n,s})$

In [9] Clements determines the minimum volume and the minimum size ideal over the set of antichains in $\Lambda_{n,s}$. Clements’ result is stated in terms of multisets. It is rephrased here for sets.

**Theorem 5.4 (Clements [9]).** Let $n, s \in \mathbb{Z}^+$ and $k \in \mathbb{N}$ be such that $\binom{n}{k} < s \leq \binom{n}{k+1}$. Let $e'$ denote the smallest of the solutions of

$$|\Delta F_{k+1} (e)| + (s - e) = \binom{n}{k}. \quad (5.2)$$

Then

(i) $V_{\min}(\Lambda_{n,s}) = e'(k + 1) + (s - e')k$,

(ii) $I_{\min}(\Lambda_{n,s}) = e' + \sum_{i=0}^{k} \binom{n}{i}$.

**Note 5.5.**

A solution to (5.2) always exists. This is shown as follows. Note that $e = s$ is a solution of $|\Delta F_{k+1} (e)| + s - e \leq \binom{n}{k}$. Let $e \in \mathbb{Z}^+$ be such that $|\Delta F_{k+1} (e)| + s - e < \binom{n}{k}$. Replacing $e$ by $e - 1$ implies that $|\Delta F_{k+1} (e - 1)| + s - e + 1 \leq \binom{n}{k}$. If there is still strict inequality, repeatedly replace $e$ by $e - 1$ until equality is achieved. Equality must be achieved since $|\Delta F_{k+1} (0)| + s > \binom{n}{k}$.

The following observation follows from Note 5.5.

**Observation 5.6.** $e' > 0$ in Theorem 5.4.
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Theorem 5.4 is interesting in the sense that, given $n$ and $s$, it shows that there exists a flat antichain in $\Lambda_{n,s}$ which simultaneously achieves minimum volume and minimum size ideal in $\Lambda_{n,s}$.

**Corollary 5.7.** Let $n, s \in \mathbb{Z}^+$. There exists an antichain $A$ in $\Lambda_{n,s}$ which is flat with $V(A) = V_{\min}(\Lambda_{n,s})$ and $|IA| = I_{\min}(\Lambda_{n,s})$.

**Proof.** Assume $n$, $s$, $k$, and $\epsilon'$ are as in the statement of Theorem 5.4. By Theorems 2.77 and 5.4 the flat collection of sets with parameters $p_i$ such that $p_i = 0$ for $i \neq k+1, k$, and $p_{k+1} = \epsilon'$, $p_k = s - \epsilon'$, is an antichain in $\Lambda_{n,s}$ which achieves minimum volume and minimum size ideal in $\Lambda_{n,s}$.

We show that there is a profile-unique antichain in $\Lambda_{n,s}$ which achieves minimum volume in $\Lambda_{n,s}$.

**Theorem 5.8.** Let $n, s \in \mathbb{Z}^+$. There is a profile-unique flat antichain in $\Lambda_{n,s}$ which achieves minimum volume in $\Lambda_{n,s}$.

Before proceeding to the proof of Theorem 5.8, we state the following corollary.

**Corollary 5.9.** Let $n, s \in \mathbb{Z}^+$. There is a profile-unique flat antichain in $\Lambda_{n,s}$ which achieves maximum volume in $\Lambda_{n,s}$.

**Proof.** Assume that Theorem 5.8 holds. Let $A$ be a profile-unique flat antichain in $\Lambda_{n,s}$ with $V(A) = V_{\min}(\Lambda_{n,s})$. $A$ exists by Theorem 5.8. Consider the complement $A'$ of $A$. By Observation 2.82, $A' \in \Lambda_{n,s}$, $V(A') = n|A| - V(A) = n|A| - V_{\min}(\Lambda_{n,s}) = V_{\max}(\Lambda_{n,s})$, and by Observation 2.83, $A'$ is flat. Since $A$ is a profile-unique antichain in $\Lambda_{n,s}$ with $V(A) = V_{\min}(\Lambda_{n,s})$, $A'$ is a profile-unique antichain in $\Lambda_{n,s}$ with $V(A') = V_{\max}(\Lambda_{n,s})$. This proves the corollary.

Theorem 5.8 directly follows from the arguments Clements used to prove Theorem 5.4. However, Theorem 5.8 is not stated explicitly in [9], so its proof is given here. The proof closely follows Clement’s line of arguments. Clements [10] acknowledges that Theorem 5.8 is an interesting addition to the results in [9]. Before proceeding to
the proof of Theorem 5.8 the following lemmas, Lemmas 5.10 and 5.11, are required. Lemma 5.10 is taken from [9] and is given without a proof.

**Lemma 5.10 (Clements [9]).** Let $A \in \Lambda_{n,s}$ be a squashed antichain. Then there exists a non-reducible and full antichain $A^* \in \Lambda_{n,s}$ with $V(A^*) \leq V(A)$.

We show that for $s$ given, there is a profile-unique antichain on $[n]$ of size $s$ which is flat, non-reducible and full.

**Lemma 5.11.** Let $n, s \in \mathbb{Z}^+$ and $k \in \mathbb{N}$ be such that $\left(\begin{array}{c} n \\ k \end{array}\right) < s \leq \left(\begin{array}{c} n \\ k+1 \end{array}\right)$. Then there is a profile-unique flat, non-reducible and full antichain $A^* \in \Lambda_{n,s}$. Further,

(i) the parameters $q_i$ of $A^*$ are such $q_i = 0$ for $i \neq k, k+1$, and $q_{k+1}$ is the smallest solution of (5.2),

(ii) $V(A^*) = V_{\min}(\Lambda_{n,s})$.

**Proof.** Let $n, s \in \mathbb{Z}^+$, $k \in \mathbb{N}$ be such that $\left(\begin{array}{c} n \\ k \end{array}\right) < s \leq \left(\begin{array}{c} n \\ k+1 \end{array}\right)$. Let $A^* \in \Lambda_{n,s}$ be a flat, non-reducible and full antichain with parameters $q_i = 0$ for $i \neq k, k+1$. Note that $k \leq \frac{n}{2}$.

As $A^*$ is full, $q_{k+1}$ is a solution of $|\Delta F_{k+1}(e)| + (s - e) = \left(\begin{array}{c} n \\ k \end{array}\right)$. As $A^*$ is non-reducible, $q_{k+1}$ is the smallest solution of $|\Delta F_{k+1}(e)| + (s - e) = \left(\begin{array}{c} n \\ k \end{array}\right)$. Thus $q_{k+1}$ is the smallest solution of (5.2) and $A^*$ exists by Note 5.5. By the uniqueness of $q_{k+1}$, there is a profile-unique flat, non-reducible and full antichain of size $s$ with parameters $q_i = 0$ for $i \neq k, k+1$.

Now, let $A$ be any flat, non-reducible and full antichain in $\Lambda_{n,s}$. Assume that it has parameters $p_i = 0$ for $i \neq l, l+1$. We show that $A \cong A^*$. If $l = k$ there is nothing to prove. Hence assume that $l \neq k$. The same argument as the one used for $q_{k+1}$ shows that $p_{l+1}$ is the smallest solution of $|\Delta F_{l+1}(e)| + (s - e) = \left(\begin{array}{c} n \\ l \end{array}\right)$. We consider two cases.

(a) $l > k$. As $A$ is non-reducible, $|\Delta F_{l+1}(p_{l+1})| < p_{l+1}$, and by Lemma 2.34, this implies that $l < \frac{n+1}{2}$. Thus $|\Delta F_{l+1}(p_{l+1})| + (s - p_{l+1}) = \left(\begin{array}{c} n \\ l \end{array}\right) < p_{l+1} + s - p_{l+1} = s$. That is, $\left(\begin{array}{c} n \\ l \end{array}\right) < \left(\begin{array}{c} n \\ k+1 \end{array}\right)$ as $s \leq \left(\begin{array}{c} n \\ k+1 \end{array}\right)$ by assumption. Given that both $k$ and $l$ are no
larger than $\frac{n}{2}$ this implies that $l \leq k$, contradicting the assumption about $l$.

(b) $l < k$. Then $l < \frac{n}{2}$ and $|\nabla L_i(s - p_{i+1})| = |\nabla A^{(i)}| \geq s - p_{i+1}$ by Lemma 2.34. Since $A$ is an antichain on $[n]$, $p_{i+1} + |\nabla A^{(i)}| \leq (\binom{n}{i+1})$. This implies that $s \leq (\binom{n}{i+1})$ and $\binom{n}{i} < (\binom{n}{i+1})$ as $s > (\binom{n}{i})$ by assumption. It follows that $l + 1 > k$ since $k \leq \frac{n}{2}$, contradicting the assumption about $l$.

We conclude that $l = k$. This implies that $A \cong A^*$. We have already shown that the parameters of $A^*$ satisfy (i). (ii) follows from (i) and Theorem 5.4.(i). This completes the proof of Lemma 5.11.

We are now in a position to prove Theorem 5.8. We prove its contrapositive.

Proof of Theorem 5.8. Let $A \in \Lambda_{n,s}$ and let $A^* \in \Lambda_{n,s}$ be the profile-unique flat, non-reducible and full antichain in Lemma 5.11. Assume that $A \not\cong A^*$. We prove that $V(A) > V(A^*)$.

By Lemma 5.11 and Observation 5.6, $A^*$ consists of $(k + 1)$-sets and possibly $k$-sets, and $V(A^*) = V_{\min}(\Lambda_{n,s})$. It follows that $V(A) \geq V(A^*)$. By Lemma 5.10 we may assume without loss of generality that $A$ is non-reducible and full. By Lemma 5.11 $A$ cannot be flat since $A \not\cong A^*$. Assume therefore that $A$ is not flat and let $b$ and $l$ respectively be the largest and smallest integer $i$ for which the parameters $p_i$ of $A$ are non-zero. Then $l < h - 1$.

We show that one can obtain $A^*$ from $A$ while strictly decreasing the volume. Let $|\nabla A^{(i)}| = |\nabla F_i([A^{(i)}])| = p$ and recall that $|A^{(i)}| = p_h$ and $|A^{(i)}| = p_l$. Two cases are considered.

(i) $p \leq p_l$. Since $p = |\nabla F_i([A^{(i)}])| = |\nabla F_i(p_h)|$ it follows that $|\nabla N F_{i-1}(p)| = |\nabla N F_{i-1}(\nabla F_i(p_h))| \geq p_h$ by Lemma 2.67.

By Corollary 2.50, $|\nabla N F_{i}(p)| \geq |\nabla N F_{i-1}(p)|$, and by Corollary 2.49, $|\nabla N C_i(p)| \geq |\nabla N F_{i}(p)|$. Let $B = F(p_h A^{(i)})$. Then $B$ is a collection $C_i(p)$. Therefore $|\nabla N B| = |\nabla N C_i(p)| \geq |\nabla N F_{i}(p)| \geq |\nabla N F_{i-1}(p)| \geq p_h$.

Thus it is possible to replace the $p_h h$-sets of $A^{(i)}$ by $p_h (l + 1)$-sets in $\nabla N B$, and the
p l-sets of \( B \) by the \( p (h - 1) \)-sets in \( \triangle A^{(h)} \). Call the newly obtained collection \( A_1 \). It is easy to verify that \( A_1 \in \Lambda_{n,s} \).

As \( A \) is assumed to be non-reducible, \(|\triangle A^{(h)}| < |A^{(h)}|\). That is, \( p < p_\ell \). We conclude that \( V(A_1) < V(A) \).

(ii) \( p > p_\ell \). By Corollary 2.50, \(|\nabla L_i(p_\ell)| \geq |\nabla L_{h-1}(p_\ell)|\), and by Corollary 2.49, \(|\nabla L_{h-1}(p_\ell)| \geq |\nabla C_{h-1}(p_\ell)|\). Let \( B = L(p_\ell, \triangle A^{(h)}) \). Then \( B \) is a collection \( C_{h-1}(p_\ell) \).

Since \( A \) is full, \(|\nabla A^{(h)}| = |\nabla L_i(p_\ell)|\). It follows that \(|\nabla A^{(h)}| = |\nabla L_i(p_\ell)| \geq |\nabla L_{h-1}(p_\ell)|\) \( \geq |\nabla C_{h-1}(p_\ell)| = |\nabla B|\). Let \( C = \nabla B \cap A^{(h)} \) and \(|C| = c\). Then \(|C| \leq |\nabla B|\) which implies that \( c \leq |\nabla A^{(h)}|\).

We replace the \( c h \)-sets in \( C \) by \( c (l+1) \)-sets in \( \nabla A^{(l)} \), and the \( pl \)-sets of \( A^{(l)} \) by the \( pl (h - 1) \)-sets of \( B \) to form a new antichain \( A_2 \in \Lambda_{n,s} \).

As \( A \) is assumed to be non-reducible, \(|\nabla C| < |C|\). Further, by taking cardinalities in Lemma 2.64, we see that \(|B| \leq |\nabla C|\). Therefore \(|B| < |C|\). That is \( p_\ell < c\). It follows that \( V(A_1) < V(A) \).

By Lemma 5.10, replacing \( A_1 \) in both Cases (i) and (ii) by a non-reducible and full antichain \( A_2 \) does not increase the volume. If \( A_2 \) is not flat then the set replacement procedure as described in (i) and (ii) is repeated with \( A_2 \) relabelled as \( A \). Eventually we must obtain the antichain \( A^* \) as the volume cannot decrease indefinitely. It follows that \( V(A) > V(A^*) \).

\[ \square \]

Note 5.12.

While replacing sets using the set replacement procedure as described in (i) and (ii) in the proof of Theorem 5.8 the ideal of the antichain does not increase. As mentioned earlier, the proof of Theorem 5.8 as given here is Essentially Clements’ proof of Theorem 5.4. One difference resides in the fact that Clements uses a variant of Lemma 5.11 to show that the parameters of \( A^* \) satisfy (5.2). It must also be noted that Clements emphasizes the fact that the volume of the antichain strictly decreases while replacing sets (one outcome being Corollary 5.13 below).

The next two corollaries are natural corollaries of Theorem 5.8. Corollary 5.13 appears
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Corollary 5.13 (Clements [9]). Let \( n, s \in \mathbb{Z}^+ \) and \( k \in \mathbb{N} \) be such that \( \binom{n}{k} < s \leq \binom{n}{k+1} \). Let \( \mathcal{A}^* \in \Lambda_{n,s} \) be such that \( V(\mathcal{A}^*) = V_{\text{min}}(\Lambda_{n,s}) \). Then, for each \( A \in \mathcal{A}^* \), \( |A| \geq k \).

Corollary 5.14. Let \( n, s \in \mathbb{Z}^+ \) and \( k \in \mathbb{N} \) be such that \( \binom{n}{k} < s \leq \binom{n}{k+1} \). Let \( \mathcal{A}^* \in \Lambda_{n,s} \) be such that \( V(\mathcal{A}^*) = V_{\text{min}}(\Lambda_{n,s}) \). Then, for each \( A \in \mathcal{A}^* \), \( |A| \leq k + 1 \).

Proof. By Theorem 5.8, \( \mathcal{A}^* \) is the antichain in Lemma 5.11. The result follows.

Theorem 5.8 can be used to show that Theorem 5.1 is a special case of Theorem 5.4, that is, Theorem 5.4 implies Theorem 5.1.

Another alternative proof for Theorem 5.1. Let \( n, k \in \mathbb{Z}^+ \). We consider two cases.

(i) Let \( \mathcal{A} \in \Lambda_{n,\binom{n}{k}} \). Then the antichain \( \mathcal{A}^* = [n]^k \) achieves \( V_{\text{min}}(\Lambda_{n,\binom{n}{k}}) = k\binom{n}{k} \) in \( \Lambda_{n,\binom{n}{k}} \) by Theorem 5.4(i). By Theorem 5.8 such an antichain is profile-unique. Therefore \( V(\mathcal{A}) > k\binom{n}{k} \) and \( |\overline{A}| > k \) for each \( \mathcal{A} \not\subseteq \mathcal{A}^* \), \( \mathcal{A} \in \Lambda_{n,\binom{n}{k}} \).

(ii) Now let \( s \) be such that \( s > \binom{n}{k} \) and let \( \mathcal{A} \in \Lambda_{n,s} \). Then \( V_{\text{min}}(\Lambda_{n,s}) > ks \) by Theorem 5.4(i) and Observation 5.6. Therefore \( V(\mathcal{A}) > ks \) and \( |\overline{A}| > k \).

In all cases \( |\overline{A}| > k \) for an antichain \( \mathcal{A} \not\subseteq [n]^k \) with \( |A| \geq \binom{n}{k} \). Note that saying that \( \mathcal{A} \not\subseteq [n]^k \) is equivalent to saying that \( \mathcal{A} \neq [n]^k \). This proves Theorem 5.1.

Theorem 5.8 also implies that if \( \mathcal{A} \in \Lambda_{n,s} \) achieves \( V_{\text{min}}(\Lambda_{n,s}) \) then \( \mathcal{A} \) also achieves \( I_{\text{min}}(\Lambda_{n,s}) \).

Corollary 5.15. Let \( \mathcal{A} \in \Lambda_{n,s} \) be an antichain with \( V(\mathcal{A}) = V_{\text{min}}(\Lambda_{n,s}) \). Then \( |\mathcal{I}\mathcal{A}| = I_{\text{min}}(\Lambda_{n,s}) \).

Proof. Let \( \mathcal{A} \) be as in the statement of the corollary. Then, by Theorems 5.4(i) and 5.8 the parameters \( p_i \) of \( \mathcal{A} \) are such that \( p_i = 0 \) for \( i \neq k + 1, k \), and \( p_{k+1} \) is the smallest solution of (5.2). It follows that \( |\mathcal{I}\mathcal{A}| = I_{\text{min}}(\Lambda_{n,s}) \) by Theorem 5.4(ii). 

The converse of Corollary 5.15 is not true in general as shown by the following example.

**Example 5.16.** Let \( n = 8, \ s = 36 \). We consider two squashed antichains \( \mathcal{A}^*, \mathcal{A} \in \mathcal{A}_{8,36} \).

Let \( \mathcal{A}^* = F_{8,3}(28) \cup \{67,18,28,38,48,58,68,78\} \). Recall that \( \mathcal{I}^{(2)}\mathcal{A}^* = \{B : B \in \mathcal{I}\mathcal{A}^*, |B| = 2\} \). Then \( |\mathcal{I}^{(2)}\mathcal{A}^*| = \binom{8}{2} \) which implies that \( \mathcal{A}^* \) is full. It can be verified that \( \mathcal{A}^* \) is non-reducible. Therefore \( V(\mathcal{A}^*) = V_{\min}(\mathcal{A}_{8,36}) \) by Lemma 5.11. It follows that \( |\mathcal{I}\mathcal{A}^*| = I_{\min}(\mathcal{A}_{8,36}) \) by Corollary 5.15. We have \( |\mathcal{I}\mathcal{A}^*| = 28 + \binom{8}{2} + \binom{8}{1} + \binom{8}{0} = 65 \) and \( V(\mathcal{A}^*) = 100 \). Therefore \( I_{\min}(\mathcal{A}_{8,36}) = 65 \) and \( V_{\min}(\mathcal{A}_{8,36}) = 100 \).

Let \( \mathcal{A} = F_{8,3}(35) \cup \{8\} \). That is, \( \mathcal{A} \) consists of the \( \binom{7}{3} \) 3-subsets of \( [7] \) followed by the set 8. Then \( |\mathcal{I}\mathcal{A}| = \binom{7}{3} + \binom{7}{2} + \binom{8}{1} + \binom{8}{0} = 65 \) and \( V(\mathcal{A}) = 106 \). But \( |\mathcal{I}\mathcal{A}| = I_{\min}(\mathcal{A}_{8,36}) \) and \( V(\mathcal{A}) > V_{\min}(\mathcal{A}_{8,36}) \).

We conclude this section by an interesting problem which Clements posed in [9].

**Open Problem 5.17 (Clements [9]).** Determine if Theorem 5.4.(i) can be derived from Theorem 5.4.(ii) and vice versa.

It is shown in Section 10.4 that the truth of the flat antichain conjecture partially solves Open Problem 5.17. That is, if the flat antichain conjecture holds, then Theorem 5.4.(ii) implies Theorem 5.4.(i) (see Corollary 10.8).

**5.4 Conclusion**

In this chapter we have seen that there is a profile-unique flat antichain \( \mathcal{A} \in \mathcal{A}_{n,s} \) which has volume \( V(\mathcal{A}) = V_{\min}(\mathcal{A}_{n,s}) \). Therefore \( V_{\min}(\mathcal{A}_{n,s}) \) is a volume which in \( \mathcal{A}_{n,s} \) is only achieved by a flat antichain. By Corollary 5.9 \( V_{\max}(\mathcal{A}_{n,s}) \) is also a volume which in \( \mathcal{A}_{n,s} \) is only achieved by a flat antichain. This suggests that there may be other volumes of antichains in \( \mathcal{A}_{n,s} \) which are only achieved by flat antichains in \( \mathcal{A}_{n,s} \).

We state the following conjecture.
Conjecture 5.18. Let \( A \in \Lambda_{n,s} \) be a flat and full antichain which has sets on two consecutive levels. Then \( A \) is a profile-unique antichain in \( \Lambda_{n,s} \) with volume \( V(A) \).

Example 5.19. Let \( n = 5 \) and \( s = 7 \). The flat antichain \( A = \{123, 124, 134, 15, 25, 35, 45\} \) on levels 3 and 2 is full. Note that \( A \in \Lambda_{5,7} \), \( V(A) = 17 \), and \( A \) is reducible. Let \( B \) be any antichain on \([5]\). If \( \{1234\} \in B \) then \( |B| \leq 5 \). If \( \{5\} \in B \) and \( B \) contains only 2-sets, then \( |B| \leq 7 \) and \( V(B) \leq 13 \). If \( \{5\} \in B \) and \( B \) contains some 4 or 3-sets, then \( |B| \leq 5 \). Thus it is not possible to construct \( B \) so that \( B \) is not flat and \( |B| = 7 \) and \( V(B) = 17 \). That is, there exists no antichain \( B \) in \( \Lambda_{5,7} \) such that \( B \) is not flat and \( A \) is a flat counterpart of \( B \). It follows that \( A \) is a profile-unique antichain in \( \Lambda_{5,7} \) with volume 17.

Conjecture 5.18 is further discussed in Chapter 9 (see Lemma 9.11). It must also be noted that there may be integers between \( V_{\min}(\Lambda_{n,s}) \) and \( V_{\max}(\Lambda_{n,s}) \) which are not the volume of any antichain in \( \Lambda_{n,s} \) (see Example 4.6). Section 10.3 addresses this issue in relation to the flat antichain conjecture.
Chapter 6

Volumes and Flat Antichains
6.1 Introduction

In this chapter we show that given any antichain $\mathcal{A}$ on $[n]$ there exists a flat antichain on $[n]$ with volume $V(\mathcal{A})$. In doing so we prove that the flat antichain conjecture holds for a given class of antichains. All results and proofs presented here arise from joint work with M. Miller and L. Branković. The main result in this chapter is

**Theorem 6.1 (with Miller and Branković [4]).** For any antichain $\mathcal{A}$ on $[n]$ there exists a flat antichain $\mathcal{A}^\ast$ on $[n]$ such that $V(\mathcal{A}^\ast) = V(\mathcal{A})$.

Theorem 6.1 is a weaker version of the flat antichain conjecture with the constraint on the size of $\mathcal{A}^\ast$ being removed. Proving Theorem 6.1 requires Theorems 6.5, 6.6 and 6.7 and Definitions 6.2 to 6.4. The definitions are given first.

**Definition 6.2.**

Let $K = \left\lceil \frac{n}{2} \right\rceil + 1$.

Definition 6.3 defines four types of antichains. The motivation for the definition is given before stating Definition 6.4.

**Definition 6.3.**

Let $\mathcal{A}$ be an antichain on $[n]$ with parameters $p_i$. Then $\mathcal{A}$ is defined to be of Type A1, A2, A3 or A4 respectively if the following conditions hold:

(i) Type A1: $n$ is odd, $p_i = 0$ for $i > K + 1$ and $i < K - 1$,

\[ 0 < p_{K+1} < K + 1 \text{ and } 0 < p_{K-1} < K; \]

(ii) Type A2: $n$ is even, $p_i = 0$ for $i > K + 1$ and $i < K - 1$,

\[ 0 < p_{K+1} < K + 1 \text{ and } p_{K-1} \neq 0; \]

(iii) Type A3: $n$ is even, $p_i = 0$ for $i > K$ and $i < K - 2$,

\[ p_K \neq 0 \text{ and } 0 < p_{K-2} < K; \]

(iv) Type A4: $n$ is even, $p_i = 0$ for $i > K + 1$ and $i < K - 2$,

\[ 0 < p_{K+1} < K + 1 \text{ and } 0 < p_{K-2} < K. \]

We next define $U_n$. It will be shown that $U_n$ is the smallest volume which cannot be achieved by an antichain on $[n]$ consisting of $K$ and $(K-1)$-sets only (see Lemma 6.11
in Section 6.2. In Theorem 6.6 it is shown that antichains on \([n]\) whose volume is at least \(U_n\) can be classified in one of the four types defined in Definition 6.3.

**Definition 6.4.**

Let \(U_n = \left(\binom{n}{K} - \frac{2n-3K+4}{2} (K-1)+1\right) K + (K-1)^2\).

We now state Theorems 6.5 to 6.7 and prove Theorem 6.1. Note that there is no reference to the size of the antichain in Theorem 6.5.

**Theorem 6.5 (with Miller and Branković [4]).** Let \(V \in \mathbb{Z}^+\). For each \(V < U_n\) there exists a flat antichain on \([n]\) with volume \(V\).

**Theorem 6.6 (with Miller and Branković [4]).** Let \(\mathcal{A}\) be a non-flat antichain on \([n]\) with \(V(\mathcal{A}) \geq U_n\). Then \(\mathcal{A}\) is of Type A1, A2, A3, or A4.

**Theorem 6.7 (with Miller and Branković [4]).** Let \(\mathcal{A}\) be an antichain on \([n]\) of Type A1, A2, A3, or A4. Then the flat antichain conjecture holds for \(\mathcal{A}\).

**Proof of Theorem 6.1.** Assume that Theorems 6.5, 6.6 and 6.7 hold. Let \(\mathcal{A}\) be an antichain on \([n]\). If \(\mathcal{A}\) is flat then there is nothing to prove. Hence assume that \(\mathcal{A}\) is not flat. If \(V(\mathcal{A}) < U_n\), then there exists a flat antichain on \([n]\) with volume \(V(\mathcal{A})\) by Theorem 6.5. If \(V(\mathcal{A}) \geq U_n\) then \(\mathcal{A}\) is of Type A1, A2, A3, or A4 by Theorem 6.6. By Theorem 6.7, \(\mathcal{A}\) has a flat counterpart which is an antichain on \([n]\). This proves Theorem 6.1.

Theorems 6.5, 6.6 and 6.7 are proved in Sections 6.2, 6.3 and 6.4 respectively. The proof of each theorem can be read independently. The chapter concludes with Section 6.5 where the significance of the volume \(U_n\) with respect to the flat antichain conjecture is discussed.

The remainder of this section presents two lemmas which relate the volume of a collection of sets \(\mathcal{B}\) to the volume of \(\triangle \mathcal{B}\) and the volume of \(\nabla \mathcal{B}\) respectively.

**Lemma 6.8.** Let \(\mathcal{B}\) be a collection of \(l\)-subsets of \([n]\). If \(l > K\) for \(n\) odd or \(l \geq K\) for \(n\) even then \(V(\triangle \mathcal{B}) \geq V(\mathcal{B})\).
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Proof. By Sperner’s lemma \(|\triangle B| \geq \frac{n-l}{n+l+1}|B|\) so that

\[ V(\triangle B) \geq (l-1) \frac{n-l}{n-l+1}|B| \geq l|B| = V(B) \]

for \(\frac{n-l}{n+l+1} \geq 1\). That is, \(V(\triangle B) \geq V(B)\) for \(l \geq \frac{n+1}{2}\). \(\square\)

Lemma 6.9. Let \(B\) be a collection of \(l\)-subsets of \([n]\). If \(l < K\) then \(V(\triangle B) \geq V(B)\).

Proof. By Sperner’s lemma \(|\nabla B| \geq \frac{n-l}{n+l+1}|B|\) so that

\[ V(\nabla B) \geq (l+1) \frac{n-l}{l+1}|B| \geq l|B| = V(B) \]

for \(n - l \geq l\). That is, \(V(\nabla B) \geq V(B)\) for \(l \leq \frac{n}{2}\). \(\square\)

6.2 All Volumes Less than \(U_n\) Can Be Achieved by a Flat Antichain

Here we prove Theorem 6.5. In order to prove that for each positive integer \(V\), \(V < U_n\), there exists a flat antichain on \([n]\) with volume \(V\) it is sufficient to construct flat antichains on \([n]\) of volume \(V\) for each \(V < U_n\). This is done in Lemmas 6.10 and 6.11 below.

Lemma 6.10. For each \(V \in \mathbb{Z}^+\), \(V \leq (K-1)^2\), there exists a flat antichain on \([n]\) with volume \(V\).

Proof. This is trivial for \(n = 1\). So assume that \(n \geq 2\). Let \(p \in \mathbb{N}\). Using only 1-sets and 2-sets, and applying Corollary 2.41, we can construct antichains on \([n]\) with volume \(V\)

\[ V = 2p + 1 \quad \text{ (by using } p \text{ 2-sets and one 1-set),} \]
\[ V = 2(p + 1) \quad \text{ (by using } (p + 1) \text{ 2-sets)} \]

for \(0 \leq p \leq \binom{n}{2} - (n - 1)\). \(\square\)

Lemma 6.11. For each \(V \in \mathbb{Z}^+\), \((K-1)^2 < V < U_n\), there exists a flat antichain on \([n]\) with volume \(V\).
**Proof.** Let \( p \in \mathbb{N} \). Using only \( K \)-sets and \((K - 1)\)-sets, and applying Corollary 2.41 and Observation 2.43, we can construct antichains on \([n]\) with volume \( V \)

\[
V = Kp + (K - 1)(K - 1) \quad \text{(by using } p \text{ \( K \)-sets and \((K - 1) \) \((K - 1)\)-sets)},
\]

\[
V = K(p + 1) + (K - 2)(K - 1) \quad \text{(by using } (p + 1) \text{ \( K \)-sets and \((K - 2) \) \((K - 1)\)-sets)},
\]

\[
V = K(p + 2) + (K - 3)(K - 1) \quad \text{(by using } (p + 2) \text{ \( K \)-sets and \((K - 3) \) \((K - 1)\)-sets)},
\]

\[
\vdots
\]

\[
V = K(p + K - 2) + (K - 1) \quad \text{(by using } (p + K - 2) \text{ \( K \)-sets and one \((K - 1)\)-set)},
\]

\[
V = K(p + K - 1) \quad \text{(by using } (p + K - 1) \text{ \( K \)-sets)}
\]

for \( 0 \leq p \leq \binom{n}{K} \). Note that \( \frac{2n - 3K + 4}{2}(K - 1) \) by Observation 2.43. All antichains constructed in this way have at most \((K - 1)\) \((K - 1)\)-sets. \( U_n \) is the first volume which cannot be obtained from only \( K \)-sets and \((K - 1)\)-sets.

Theorem 6.5 follows immediately.

**Proof of Theorem 6.5.** The theorem follows from Lemmas 6.10 and 6.11.

### 6.3 Antichains with Volume \( V \geq U_n \)

Theorem 6.6 is proved in this section. The aim here is to find a sufficient condition for an antichain \( \mathcal{A} \) on \([n]\) to be of Type A1, A2, A3 or A4 and to show that this condition is that the volume of \( \mathcal{A} \) be at least \( U_n \). To do so we need to find sufficient conditions for the volume of an antichain on \([n]\) to be less than \( U_n \). This is done in Lemmas 6.12 to 6.18. Lemmas 6.12 to 6.16 together with Lemma 6.17 determine some antichains whose volume is less than \( U_n \). Lemma 6.18 summarises these results and states sufficient conditions for the volume of an antichain on \([n]\) to be less than \( U_n \). The proof of Theorem 6.6 shows that these sufficient conditions are that the antichain is *not* of Type A1, A2, A3 or A4. The proof of Theorem 6.6 is given after the proof of Lemma 6.18.
The first three lemmas give an upper bound for $V(A)$ for an antichain $A$ satisfying various conditions. Lemma 6.12 deals with antichains containing no sets larger than $K + 1$. Lemma 6.13 considers antichains containing no sets smaller than $K - 1$ for $n$ odd, and Lemma 6.14 considers antichains containing no sets smaller than $K - 2$ for $n$ even.

**Terminology.** The expressions **first** and **last** will be used to refer to collections of sets at the beginning and end of a collection of sets in squashed order. This requires the universal set to be defined.

**Lemma 6.12.** Let $A$ be a squashed antichain on $[n]$ with parameters $p_i$ such that $p_i = 0$ for $i > K + 1$ and $p_{K+1} \geq K + 2$. Let $A^*$ be the antichain obtained from $A$ by replacing all but the first $(K + 2)$ $(K + 1)$-sets of $A$ by all the $K$-sets in their new-shadow. Then $V(A) \leq V(A^*)$.

**Proof.** We begin by describing the list of the $\binom{n}{K+1}$ $(K + 1)$-sets in squashed order. The idea of the squashed order is to use as few elements as possible when listing the sets. The first $(K + 1)$-set is $\{1, \ldots, K + 1\}$. This set is followed by $\binom{K+1}{K}$ sets, each of them being the union of one of the $\binom{K+1}{K}$ $K$-subsets of $[K + 1]$ with the set $\{K + 2\}$. This collection is itself followed by $\binom{K+2}{K}$ sets, each of them being the union of one of the $\binom{K+2}{K}$ $K$-subsets of $[K + 2]$ with the set $\{K + 3\}$.

Therefore the collection of $\binom{n}{K+1}$ $(K + 1)$-sets in squashed order can be partitioned into subcollections of size $\binom{i}{K}$ each, $K \leq i \leq n - 1$. For a given $i$, any set in the subcollection of size $\binom{i}{K}$ is the union of one of the $\binom{i}{K}$ $K$-subsets of $[i]$ and $\{i + 1\}$.

Let $A$ be as defined in the statement of the lemma and consider $A^{(K+1)}$, the subcollection of $A$ consisting of its $(K + 1)$-sets. By the introductory remark, $A^{(K+1)}$ is the union of consecutive collections $A_i$ of consecutive $(K + 1)$-sets in squashed order where for some $I$, $K \leq I \leq n - 1$, $|A_i| = \binom{i}{K}$ for $K \leq i < I$, $|A_i| \leq \binom{i}{K}$ for $i = I$, and $|A_i| = 0$ for $i > I$.

Let $i$ be given, $K \leq i \leq I$. We have seen that any $(K + 1)$-set of $A_i$ is the union of one of the $\binom{i}{K}$ $K$-subsets of $[i]$ and $\{i + 1\}$. Let $B_i = \{B : |B| = K, B \subset A, A \in A_i, i + 1 \notin B\}$.
Then $A_i = B_i \sqcup \{i + 1\}$. If $i = K$ then $A_K = F_{K+1}(1)$ and $|\triangle_N A_K| > |\triangle_N B_K|$ by Corollary 2.41. If $i$ is such that $K < i \leq I$ then $|\triangle_N A_i| = |\triangle_N B_i|$ by Lemma 2.30.

Since $\mathcal{A}$ is squashed, the sets in $B_i$ constitute an initial segment of $K$-subsets of $[i]$ in squashed order. This implies that $|\triangle_N B_i| = |\triangle_N A_i|$ by Observation 2.9. It follows that

$$|\triangle_N A_i| \geq |\triangle B_i|. \quad (6.1)$$

Moreover,

$$|\triangle B_i| \geq \frac{K}{i - K + 1} |B_i| \quad (6.2)$$

by Sperner’s lemma. Note that $|B_i| = |A_i|$. Therefore, combining (6.1) and (6.2) gives

$$|\triangle N A_i| \geq \frac{K}{i - K + 1} |A_i|. \quad (6.3)$$

Note that (6.1), (6.2) and (6.3) hold for each value of $i$, $K \leq i \leq I$. Thus,

$$V(\triangle N A_i) = K |\triangle N A_i| \geq K \frac{K}{i - K + 1} |A_i| \quad \text{(by (6.3))}$$

$$> (K + 1) |A_i| \quad \text{(for } K \leq i \leq n - 1)$$

$$= V(A_i)$$

so that

$$V(\triangle N A_i) > V(A_i) \quad \text{for } K \leq i \leq n - 1. \quad (6.4)$$

Also,

$$\left|\triangle N (A^{(K+1)} \setminus (A_K \cup A_{K+1}))\right| = \sum_{i=K+2}^{I} |\triangle N A_i|$$
by Observation 2.10, so

\[ V \left( \Delta_N \left( A^{(K+1)} \setminus (A_K \cup A_{K+1}) \right) \right) = \sum_{i=K+2}^{I} V (\Delta_N A_i) \]

\[ > \sum_{i=K+2}^{I} V (A_i) \]

(by (6.4))

\[ = V \left( A^{(K+1)} \setminus (A_K \cup A_{K+1}) \right). \quad (6.5) \]

\( A^* \) is the antichain obtained from \( A \) by replacing all the \((K + 1)\)-sets in \( A_i \), \( K + 2 \leq i \leq I \), by all the \( K \)-sets in \( \Delta_N A_i \). Note that \( |A_K \cup A_{K+1}| = K + 2 \). Then

\[ V (A^*) = \\
\[ V (A) - V \left( A^{(K+1)} \setminus (A_K \cup A_{K+1}) \right) + V \left( \left( A^{(K+1)} \setminus (A_K \cup A_{K+1}) \right) \setminus \Delta_N \left( A^{(K+1)} \setminus (A_K \cup A_{K+1}) \right) \right). \]

It follows that

\[ V (A^*) \geq V (A) \]

by (6.5). Equality holds if and only if \( A \) has exactly \((K + 2)\) \((K + 1)\)-sets. This proves the lemma.

We now need to establish a similar result to the one stated in the previous lemma in the case when \( A^* \) is obtained from \( A \) by replacing sets in \( A \) by sets in their new-shade. The cases \( n \) odd and \( n \) even are considered in Lemmas 6.13 and 6.14 respectively. The proofs of these lemmas are similar to the proof of Lemma 6.12 and some details have been omitted.

**Lemma 6.13.** Let \( n \) be odd. Let \( A \) be an antichain on \([n]\) with parameters \( p_i \) such that \( p_i = 0 \) for \( i < K - 1 \) and \( p_{K-1} \geq K + 1 \). Assume that the \((K - 1)\)-sets of \( A \) form a terminal segment of \((K - 1)\)-subsets of \([n]\) in squashed order. Let \( A^* \) be the antichain obtained from \( A \) by replacing all but the last \((K + 1)\) \((K - 1)\)-sets of \( A \) by all the \( K \)-sets in their new-shade. Then \( V(A) \leq V(A^*) \).

**Proof.** Let \( A \) be as defined in the statement of the lemma. Note that \( n = 2K - 1 \) as \( n \) is odd. Let \( i, I \in \mathbb{Z}^+ \) be such that \( K - 1 \leq i, I \leq n - 1 \). Then there exists an \( I \) such
that \(\mathcal{A}^{(K-1)}\) is the union of consecutive collections \(\mathcal{A}_i\) of consecutive \((K-1)\)-sets in antilexicographic order, where \(|\mathcal{A}_i| = \binom{i}{i - K + 1}\) for \(K - 1 \leq i < I\), \(|\mathcal{A}_i| \leq \binom{i}{i - K + 1}\) for \(i = I\), and \(|\mathcal{A}_i| = 0\) for \(i > I\).

To see this, note that the last \((K-1)\)-set in squashed order is the set \(\{K + 1, K + 2, \ldots, n\}\). This set is preceded by \(\binom{K}{1}\) \((K-1)\)-sets, each of them being the union of one of the singletons of \([K]\) with the set \(\{K + 2, \ldots, n\}\). This collection is itself preceded by \(\binom{K+1}{2}\) \((K-1)\)-sets, each of them being the union of one of the \(\binom{K+1}{2}\) \(2\)-sets with the set \(\{K + 3, \ldots, n\}\).

In general, for \(i\) given, \(K - 1 \leq i \leq I\), any \((K-1)\)-set of \(\mathcal{A}_i\) is the union of one of the \(\binom{i}{i - K + 1}\) \((i - K + 1)\)-sets and \(\{i+2, \ldots, n\}\). Let \(\mathcal{B}_i = \{B : |B| = i - K + 1, B \subset A, A \in \mathcal{A}_i, B \cap \{i+2, \ldots, n\} = \emptyset\}\). Then \(\mathcal{A}_i = \mathcal{B}_i \cup \{i+2, \ldots, n\}\) and \(|\nabla_N \mathcal{A}_i| = |\nabla_N \mathcal{B}_i|\) by Lemma 2.30.

The sets in \(\mathcal{B}_i\) constitute a terminal segment of \((i - K + 1)\)-subsets of \([i]\) in squashed order, so that \(|\nabla_N \mathcal{B}_i| = |\nabla \mathcal{B}_i|\) by Observation 2.9. It follows that

\[
|\nabla_N \mathcal{A}_i| = |\nabla \mathcal{B}_i|.
\] (6.6)

Moreover,

\[
|\nabla \mathcal{B}_i| \geq \frac{K - 1}{i - K + 2} |\mathcal{B}_i|
\] (6.7)

by Sperner’s lemma. Note that \(|\mathcal{B}_i| = |\mathcal{A}_i|\). Therefore, combining (6.1) and (6.7) gives

\[
|\nabla_N \mathcal{A}_i| \geq \frac{K - 1}{i - K + 2} |\mathcal{A}_i|.
\] (6.8)

Note that (6.6), (6.7) and (6.8) hold for each value of \(i, K - 1 \leq i \leq I\). Thus,

\[
V(\nabla_N \mathcal{A}_i) = K|\nabla_N \mathcal{A}_i|
\geq K \cdot \frac{K - 1}{i - K + 2} |\mathcal{A}_i|
\geq (K - 1) |\mathcal{A}_i|
\geq V(\mathcal{A}_i)
\] (for \(K - 1 \leq i \leq I \leq n - 1\))
so that
\[ V(\nabla_N A_i) \geq V(A_i) \quad \text{for } K - 1 \leq i \leq n - 1. \] (6.9)

Also,
\[ \left| \nabla_N \left( A^{(K-1)} \setminus (A_{K-1} \cup A_K) \right) \right| = \sum_{i=K+1}^{I} |\nabla_N A_i| \]
by Observation 2.10, so
\[
V \left( \nabla_N \left( A^{(K-1)} \setminus (A_{K-1} \cup A_K) \right) \right) = \sum_{i=K+1}^{I} V(\nabla_N A_i)
\geq \sum_{i=K+1}^{I} V(A_i)
\quad \text{(by (6.9))}
= V \left( A^{(K-1)} \setminus (A_{K-1} \cup A_K) \right).
\]

Lemma 6.14. Let \( n \) be even. Let \( A \) be an antichain on \([n]\) with parameters \( p_i \) such that \( p_i = 0 \) for \( i < K - 2 \) and \( p_{K-2} \geq K + 1 \). Assume that the \((K-2)\)-sets of \( A \) form a terminal segment of \((K-2)\)-subsets of \([n]\) in squashed order. Let \( A^* \) be the antichain obtained from \( A \) by replacing all but the last \((K+1)\) \((K-2)\)-sets of \( A \) by all the \((K-1)\)-sets in their new-shade. Then \( V(A) \leq V(A^*) \).

Proof. Let \( A \) be as defined in the statement of the lemma. Note that \( n = 2K - 2 \) as \( n \) is even. Let \( i, I \in \mathbb{Z}^+ \) such that \( K - 1 \leq i, I \leq n - 1 \). Then there exists an \( I \) such that \( A^{(K-1)} \) is the union of consecutive collections \( A_i \) of consecutive \((K-2)\)-sets in antilexicographic order, where \( |A_i| = \binom{i}{-K+1} \) for \( K - 1 \leq i < I \), \( |A_i| \leq \binom{i}{-K+1} \) for \( i = I \), and \( |A_i| = 0 \) for \( i > I \).

The same argument as the one used in the proof of Lemma 6.13 shows that
\[ V \left( \nabla_N \left( A^{(K-2)} \setminus (A_{K-1} \cup A_K) \right) \right) \geq V \left( A^{(K-2)} \setminus (A_{K-1} \cup A_K) \right). \] \( \square \)
We can say more about the volumes of the antichains considered in the three previous lemmas.

**Lemma 6.15.** Let $\mathcal{A}^*$ be the antichain on $[n]$ in one of the Lemmas 6.12, 6.13 or 6.14.

(i) Let $\mathcal{A}^*$ be as in Lemma 6.12.

Then $V(\mathcal{A}^*) \leq ((\binom{n}{K} - \binom{K+2}{2}) K + (K + 2)(K + 1)$.

(ii) Let $\mathcal{A}^*$ be as in Lemma 6.13.

Then $V(\mathcal{A}^*) \leq ((\binom{n}{K} - \binom{n-K+2}{2}) K + (K + 1)(K - 1)$.

(iii) Let $\mathcal{A}^*$ be as in Lemma 6.14.

Then $V(\mathcal{A}^*) \leq ((\binom{n}{K} - \binom{n-K+3}{2}) (K - 1) + (K + 1)(K - 2)$.

**Proof.** In each case, let $\mathcal{A}^*$ be as in the statement of the lemma. Recall that in (i) $\mathcal{A}^*$ is squashed, and that in (ii) and (iii) the sets of smallest size $l$ form a terminal segment of $l$-subsets of $[n]$ in squashed order. Each case is considered in turn. Assume that $\mathcal{A}^*$ has parameters $p_i$.

(i) In this case, $p_i = 0$ for $i > K + 1$ and $p_{K+1} = K + 2$. Let $\mathcal{A}_1$ be the antichain obtained from $\mathcal{A}^*$ by replacing all the sets of size at most $K$ by the sets in $\mathcal{T}^{(K)} \mathcal{A}^*$. Then $V(\mathcal{A}^*) \leq V(\mathcal{A}_1)$ by repeated applications of Lemma 6.9. Observe that $\mathcal{A}_1$ consists of $(K + 2) (K + 1)$-sets and some $K$-sets. Also, $|\Delta F_{K+1}(K + 2)| = \binom{K+2}{2}$ by Corollary 2.41. The result follows.

(ii) In this case $n$ is odd and $p_i = 0$ for $i < K - 1$ and $p_{K-1} = K + 1$. Let $\mathcal{A}_1$ be the antichain obtained from $\mathcal{A}^*$ by replacing all the sets of size at least $K$ by the sets in $\mathcal{T}^{(K)} \mathcal{A}^*$. Then $V(\mathcal{A}^*) \leq V(\mathcal{A}_1)$ by repeated applications of Lemma 6.8. Observe that $\mathcal{A}_1$ consists of $(K + 1) (K - 1)$-sets and some $K$-sets. Also, $|\nabla L_{K-1}(K + 1)| = \binom{n-K+2}{2}$ by Corollary 2.41. The result follows.

(iii) Here $n$ is even and $p_i = 0$ for $i < K - 2$ and $p_{K-1} = K + 1$. Using a similar argument to the one used in (ii) one obtains an antichain $\mathcal{A}_1$ consisting of $(K + 1) (K - 2)$-sets and some $(K - 1)$-sets such that $V(\mathcal{A}^*) \leq V(\mathcal{A}_1)$. The result follows by observing that $|\nabla L_{K-2}(K + 1)| = \binom{n-K+3}{2}$ by Corollary 2.41. \qed
The upper bounds on volumes in Lemma 6.15 have been established for the antichains \( \mathcal{A}^* \) as defined in Lemmas 6.12, 6.13 or 6.14. The next lemma shows that the same bounds also hold for three other types of antichains.

**Lemma 6.16.** Let \( \mathcal{A} \) be an antichain on \([n]\) with parameters \( p_i \).

(i) Let \( p_i = 0 \) for \( i > K + 1 \) and \( p_{K+1} = K + 1 \).

Then \( V(\mathcal{A}) \leq \binom{n}{K} - \binom{K^2}{2} K + (K + 2)(K + 1) \).

(ii) Let \( n \) be odd and let \( p_i = 0 \) for \( i < K - 1 \) and \( p_{K-1} = K \).

Then \( V(\mathcal{A}) \leq \binom{n}{K} - \left(\binom{n-K}{2}\right) K + (K + 1)(K - 1) \).

(iii) Let \( n \) be even and let \( p_i = 0 \) for \( i < K - 2 \) and \( p_{K-2} = K \).

Then \( V(\mathcal{A}^*) \leq \left(\binom{n}{K} - \binom{n-K^2}{2}\right) (K - 1) + (K + 1)(K - 2) \).

**Proof.** First note that \( \mathcal{A} \) can either be assumed to be squashed by Theorem 2.78, or that its sets of smallest size \( l \) can be assumed to form a terminal segment of \( l \)-subsets of \([n]\) in squashed order by Corollary 2.37. In each case the proof is similar to the proof of the corresponding case in Lemma 6.15 and so it is not repeated. It suffices to observe that, by Corollary 2.41, \( |\Delta F_{K+1}(K + 1)| = \binom{K-2}{2} \) in Case (i), \( |\nabla L_{K-1}(K)| = \binom{n-K-2}{2} \) in Case (ii), and \( |\nabla L_{K-2}(K)| = \binom{n-K^3}{2} \) in Case (iii). \( \square \)

It turns out that the three upper bounds given in Lemmas 6.15 and 6.16 are smaller than \( U_n \).

**Lemma 6.17.**

(i) \( \binom{n}{K} - \binom{K^2}{2} K + (K + 2)(K + 1) < U_n \),

(ii) \( \binom{n}{K} - \left(\binom{n-K}{2}\right) K + (K + 1)(K - 1) < U_n \) for \( n \) odd,

(iii) \( \left(\binom{n}{K-1} - \binom{n-K^3}{2}\right) (K - 1) + (K + 1)(K - 2) < U_n \) for \( n \) even.

**Proof.** Each inequality requires a simple algebraic derivation whose details are left to the reader. \( \square \)

It is now possible to describe some antichains whose volume is smaller than \( U_n \). This
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is the object of the next lemma.

**Lemma 6.18.** Let \( A \) be an antichain on \([n]\).

(i) Let \(|\mathcal{I}^{(K+1)}A| \geq K + 1\). Then \( V(A) < U_n \).

(ii) Let \( n \) be odd and let \(|\mathcal{F}^{(K-1)}A| \geq K\). Then \( V(A) < U_n \).

(iii) Let \( n \) be even and let \(|\mathcal{F}^{(K-2)}A| \geq K\). Then \( V(A) < U_n \).

**Proof.** Let \( A \) be as in the statement of the lemma. Each case is considered in turn.

(i) Let \( A_1 \) be the antichain obtained from \( A \) by replacing all the sets of size at least \( K + 1 \) by all the sets in \( \mathcal{I}^{(K+1)}A \). Then \( V(A) \leq V(A_1) \) by repeated applications of Lemma 6.8. Note that \(|\mathcal{I}^{(K+1)}A_1| = |\mathcal{I}^{(K+1)}A| \geq K + 1\) and that \( A_1 \) does not contain any sets larger than \( K + 1 \). If \(|\mathcal{I}^{(K+1)}A_1| = K + 1\) then \( V(A_1) < U_n \) by Lemmas 6.16.(i) and 6.17.(i). If \(|\mathcal{I}^{(K+1)}A_1| > K + 1\) then \( V(A_1) < U_n \) by Lemmas 6.12, 6.15.(i) and 6.17.(i).

(ii) Let \( A_1 \) be the antichain obtained from \( A \) by replacing all the sets of size at most \( K - 1 \) by all the sets in \( \mathcal{F}^{(K-1)}A \). Then \( V(A) \leq V(A_1) \) by repeated applications of Lemma 6.9. Note that \(|\mathcal{F}^{(K-1)}A_1| = |\mathcal{F}^{(K-1)}A| \geq K\) and that \( A_1 \) does not contain any sets smaller than \( K - 1 \). If \(|\mathcal{F}^{(K-1)}A_1| = K\) then \( V(A_1) < U_n \) by Lemmas 6.16.(ii) and 6.17.(ii). If \(|\mathcal{F}^{(K-1)}A_1| > K\) then \( V(A_1) < U_n \) by Lemmas 6.13, 6.15.(ii) and 6.17.(ii).

(iii) The proof is very similar to the proof of (ii). Replace \((K - 1)\)-sets by \((K - 2)\)-sets and use Lemmas 6.16.(iii), 6.14 and 6.17.(iii) in lieu of Lemmas 6.16.(ii), 6.13 and 6.17(ii) respectively. \(\square\)

We are now ready to characterise those non-flat antichains whose volume is no smaller than \( U_n \). We prove Theorem 6.6.

**Proof of Theorem 6.6.** Let \( A \) be a non-flat antichain on \([n]\) with volume \( V(A) \geq U_n \).

By Lemma 6.18, \(|\mathcal{I}^{(K+1)}A| < K + 1\), and \(|\mathcal{F}^{(K-1)}A| < K\) for \( n \) odd, or \(|\mathcal{F}^{(K-2)}A| < K\) for \( n \) even. Recall that \( n = 2K' - 1 \) when \( n \) is odd and that \( n = 2K' - 2 \) when \( n \)
is even. By Observation 2.44 it follows that \( p_i = 0 \) for \( i > K + 1 \) and \( p_i = 0 \) for \( i < K - 1 \) and \( n \) odd, or \( p_i = 0 \) for \( i < K - 2 \) and \( n \) even. Given that \( A \) is not flat, \( A \) must be of Type A1, A2, A3 or A4. This proves Theorem 6.6.

### 6.4 A Special Case of the Flat Antichain Conjecture

In this section we prove Theorem 6.7. This requires Lemmas 6.19 to 6.21. The three lemmas show that the flat antichain conjecture holds for antichains of Type A1, A2, A3 and A4. Note the implicit use of Theorem 2.36 and Corollary 2.37 in the proofs of Lemmas 6.19, 6.20 and 6.21 whereby the sets of larger size \( h \) are assumed to form an initial segment of \( h \)-sets in squashed order and the sets of smallest size \( l \) are assumed to form a terminal segment of \( l \)-subsets of \( [n] \) in squashed order.

**Lemma 6.19.** Let \( A \) be an antichain on \( [n] \) of Type A1 or A2. Then there exists a flat antichain on \( [n] \) of size \( |A| \) and volume \( V(A) \).

**Proof.** Let \( p = \min \{ p_{K+1}, p_{K-1} \} \). Then \( p \leq p_{K+1} < K+1 \) and \( \Delta_N L(p, F_{K+1}(p_{K+1})) \geq 2p \) by Observation 2.42(i). It follows that there exists a flat antichain on \( [n] \) of size \( |A| \) and volume \( V(A) \) with parameters \( q_i = 0 \) for \( i > K + 1 \) and \( i < K - 1 \), \( q_{K+1} = p_{K+1} - p \), \( q_K = p_K + 2p \), \( q_{K-1} = p_{K-1} - p \). \( \square \)

**Lemma 6.20.** Let \( A \) be an antichain on \( [n] \) of Type A3. Then there exists a flat antichain on \( [n] \) of size \( |A| \) and volume \( V(A) \).

**Proof.** Note that \( n = 2K - 2 \) and let \( p = \min \{ p_K, p_{K-2} \} \). Then \( p \leq p_{K-2} < K = n - (K - 2) \) and \( |\nabla F(p, L_{K-2}(p_{K-2}))| \geq 2p \) by Observation 2.42(ii). It follows that there exists a flat antichain on \( [n] \) of size \( |A| \) and volume \( V(A) \) with parameters \( q_i = 0 \) for \( i > K \) and \( i < K - 2 \), \( q_K = p_K - p \), \( q_{K-1} = p_{K-1} + 2p \), \( q_{K-2} = p_{K-2} - p \). \( \square \)

**Lemma 6.21.** Let \( A \) be an antichain on \( [n] \) of Type A4. Then there exists a flat antichain on \( [n] \) of size \( |A| \) and volume \( V(A) \).
Proof. Note that \( n = 2K - 2 \) and let \( p = \min\{p_{K+1}, p_{K-2}\} \). Then \( p \leq p_{K+1} < K + 1 \) and \( p \leq p_{K-2} < K = n - (K - 2) \). By Observations 2.42(i) and (ii) it follows that \( |\Delta NL(p, F_{K+1}(p_{K+1}))| > 2p > p \) and \( |\nabla LF(p, L_{K-2}(p_{K-2}))| > 2p > p \). Therefore there exists an antichain on \([n]\) of size \( |\mathcal{A}| \) and volume \( V(\mathcal{A}) \) with parameters \( q_i = 0 \) for \( i > K + 1 \) and \( i < K - 2 \), \( q_{K-1} = p_{K+1} - p \), \( q_K = p_{K} + p \), \( q_{K-1} = p_{K-1} + p \), \( q_{K-2} = p_{K-2} - p \).

It is easy to see that \( \mathcal{A} \) is either flat or is of Type A2 or A3. In the latter case, there exists a flat antichain on \([n]\) of size \( |\mathcal{A}| \) and volume \( V(\mathcal{A}) \) by Lemmas 6.19 and 6.20.

The proof of Theorem 6.7 now follows.

Proof of Theorem 6.7. Lemmas 6.19 to 6.21 show that the flat antichain conjecture holds for antichains on \([n]\) of Type A1, A2, A3, and A4.

6.5 Conclusion: \( U_n \) and the Flat Antichain Conjecture

Combining Theorems 6.6 and 6.7 enables us to give an alternative statement of Theorem 6.7.

Theorem 6.22. Let \( \mathcal{A} \) be an antichain on \([n]\) with \( V(\mathcal{A}) \geq U_n \). Then the flat antichain conjecture holds for \( \mathcal{A} \).

Proof. Let \( \mathcal{A} \) be an antichain with \( V(\mathcal{A}) \geq U_n \). If \( \mathcal{A} \) is flat then there is nothing to prove. If \( \mathcal{A} \) is not flat then \( \mathcal{A} \) is of Type A1, A2, A3, or A4 by Theorem 6.6 and the flat antichain conjecture holds for \( \mathcal{A} \) by Theorem 6.7.

It is of interest to know how large is \( U_n \). Lemma 6.25 below shows that \( U_n \) is bounded by \( V_{\max2}(A_n) \) and \( V_{\max}(A_n) \). We begin by expressing \( V_{\max}(A_n) \) and \( V_{\max2}(A_n) \) in terms of \( K \). This is done in Lemma 6.23 and Observation 6.24.

Lemma 6.23 (Maire [23]). For odd, \( V_{\max}(A_n) = K \binom{n}{K} \). For even, \( V_{\max}(A_n) = K \binom{n}{K}(K-1) \binom{n}{K-1} \).
Observation 6.24. For $n$ odd, $V_{max}(\Lambda_n) = (K + 1) \binom{n}{K+1} = (K - 1) \binom{n}{K-1}$. For $n$ even, $V_{max}(\Lambda_n) = (K + 1) \binom{n}{K+1} = (K - 2) \binom{n}{K-2}$.

Lemma 6.25. $V_{max}(\Lambda_n) < U_n < V_{max}(\Lambda_n)$.

Proof. As for the proof of Lemma 6.17 each inequality can be shown to hold by simple algebraic derivations whose details are left to the reader. \qed

Example 6.26. Values of $U_n$, $V_{max}(\Lambda_n)$ and $V_{max}^2(\Lambda_n)$ are given for some values of $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{max}^2(\Lambda_n)$</td>
<td>20</td>
<td>30</td>
<td>105</td>
<td>168</td>
</tr>
<tr>
<td>$U_n$</td>
<td>22</td>
<td>49</td>
<td>117</td>
<td>251</td>
</tr>
<tr>
<td>$V_{max}(\Lambda_n)$</td>
<td>30</td>
<td>60</td>
<td>140</td>
<td>280</td>
</tr>
</tbody>
</table>

Theorem 6.22 is interesting as it provides a simple way of describing a class of antichains for which the flat antichain conjecture holds. However it can be seen from Lemma 6.25 that the class of antichains concerned is a small subset of $\Lambda_n$. 
Chapter 7

On New-Shadows and Shades:
The 3-Levels Result
Chapter 7. On New-Shadows and Shades: The 3-Levels Result

7.1 The 3-Levels Result: Theorem 7.1

This chapter states and proves Theorem 7.1 which is called the 3-levels result as it relates appropriately chosen collections of \((k + 1)\)-sets and \((k - 1)\)-sets. Theorem 7.1 provides a simple lower bound in terms of \(p\) for the expressions \(|\triangle_N L_{n,k+1}(p)| + |\nabla_N L_{n,k-1}(p)|\) and \(|\triangle_N F_{n,k+1}(p)| + |\nabla_N F_{n,k-1}(p)|\). We have seen in Chapter 2 that Corollary 2.58 gives an exact value for \(|\triangle_N L_{n,k+1}(p)|\) and that Theorem 2.59 gives a lower bound for \(|\triangle_N L_{n,k+1}(p)|\). These results are often difficult to apply in an analytic sense. An advantage of the lower bound on \(|\triangle_N L_{n,k+1}(p)|\) as given by Theorem 7.1 is that it is easier to apply, even though it is expressed in terms of \(|\nabla_N L_{n,k-1}(p)|\) and \(p\). For example, Theorem 7.1 enables us to prove that the flat antichain conjecture holds in six special cases as will be shown in Chapter 8. It is not at all clear that Corollary 2.58 or Theorem 2.59 could be used for the same purpose.

After stating Theorem 7.1, we show that proving Theorem 7.1.(i) is sufficient to prove Theorem 7.1 in its entirety. The proof of Theorem 7.1.(i) is long and is deferred until Sections 7.4, 7.5 and 7.6. The outline of the proof can be found in Section 7.3. Additional lemmas which are needed in the proof of Theorem 7.1.(i) are given in Section 7.2 and Appendix E. Appendix E contains results concerning algebraic inequalities.

The main result in this chapter is

**Theorem 7.1 (The 3-levels result).** Let \(n, k \in \mathbb{Z}^+\) be such that \(1 \leq k < n\). Then, for \(0 \leq p \leq \min\left\{\binom{n}{k+1}, \binom{n}{k-1}\right\}\),

(i) \(|\triangle_N L_{n,k+1}(p)| + |\nabla_N L_{n,k-1}(p)| \geq 2p\), and

(ii) \(|\triangle_N F_{n,k+1}(p)| + |\nabla_N F_{n,k-1}(p)| \geq 2p\).

Equality holds if and only if \(p = 0\).
Note 7.2

Theorem 7.1 trivially holds for $p = 0$ so when proving Theorem 7.1 only values of $p$ in $\mathbb{Z}^+$ are considered. We have chosen to state Theorem 7.1 in this form for convenience, as there are cases where the inequality need not be strict when applying Theorem 7.1, thus removing the need to specify that $p > 0$. This is the case when proving some results in Chapter 8.

Further, using Observation 2.9, the expressions (i) and (ii) of Theorem 7.1 can be rewritten as

$$|\triangle N L_{n,k+1}(p)| + |\nabla L_{n,k-1}(p)| \geq 2p,$$

and

$$|\triangle F_{n,k+1}(p)| + |\nabla N F_{n,k-1}(p)| \geq 2p$$

respectively.

We show that to prove Theorem 7.1 one only needs to prove one inequality.

Lemma 7.3. Theorem 7.1.(i) holds if and only if Theorem 7.1.(ii) holds.

Proof. Replacing $k$ by $n - k$ in Theorems 7.1.(i) and (ii) and applying Lemmas 2.18 and 2.22 together with Observation 2.9 gives Theorems 7.1.(ii) and (i) respectively. \hfill \square

Proving Theorem 7.1.(i) is the object of the remainder of the chapter. Recall that if two collections $A$ and $B$ correspond then $|A| = |B|$, and $|\triangle N A| = |\triangle N B|$ and $|\nabla N A| = |\nabla N B|$.

7.2 Additional Lemmas

The following results are needed in the proof of Theorem 7.1.(i). All results assume that $n$ and $k$ are positive integers with $1 \leq k < n$. The lemmas in this section provide useful equalities and inequalities relating the sizes of the new-shadows and the new-shades of some collections of sets. Lemmas 7.4 to 7.10 make use of the fact that the
collections of sets considered correspond. The ordering on sets is assumed to be the squashed order.

**Lemma 7.4.**

\[
\bigl| \triangle_n L_{n,k} \left( \binom{n-1}{k-1} + p \right) \bigr| = \bigl| \triangle_n L_{n-1,k-1} \left( \binom{n-1}{k-1} \right) \bigr| + \bigl| \triangle_n L_{n-1,k} (p) \bigr|, \quad \text{and}
\]

\[
\bigl| \nabla_n L_{n,k} \left( \binom{n-1}{k-1} + p \right) \bigr| = \bigl| \nabla_n L_{n-1,k-1} \left( \binom{n-1}{k-1} \right) \bigr| + \bigl| \nabla_n L_{n-1,k} (p) \bigr|.
\]

**Proof.** The collection of \( \binom{n-1}{k} \) \( k \)-sets in squashed order consists of the \( \binom{n-1}{k} \) \( k \)-subsets of \([n-1]\) followed by the \( \binom{n-1}{k-1} \) \( k \)-subsets of \([n]\) which contain the element \( n \). Thus the \( p \) consecutive \( k \)-sets which come before \( L_{n,k} \left( \binom{n-1}{k-1} \right) \) are \( k \)-subsets of \([n-1]\). Lemma 7.4 then follows from Lemma 2.31 and Observation 2.10. \( \square \)

**Example 7.5.** For \( n = 5, k = 3 \) and \( p = 2 \), \( |\triangle_N L_{5,3} \left( \binom{4}{2} + 2 \right) | = |\triangle_N \{134, 234, 125, 135, 235, 145, 245, 345\}| + |\triangle_N \{134, 234\}| = |\triangle_N \{L_{4,2} \left( \binom{4}{2} \right) \cup \{5\}\}| + |\triangle_N L_{4,3} (2) | = |\triangle_N L_{4,2} \left( \binom{4}{2} \right) | + |\triangle_N L_{4,3} (2) |. \quad \circ \)

**Lemma 7.6.** If \( p_1 \geq \binom{n-1}{k-1} \), then

\[
|\triangle_N L_{n,k} (p_1 + p_2) | \geq |\triangle_N L_{n,k} (p_1) | + |\triangle_N L_{n-1,k} (p_2) |.
\]

**Proof.** If \( p_1 \geq \binom{n-1}{k-1} \) the \( p_2 \) consecutive \( k \)-sets that come before \( L_{n,k} (p_1) \) are \( k \)-sets on \([n-1]\). Therefore these \( p_2 \) \( k \)-sets form a collection \( C_{n-1,k} (p_2) \) that comes immediately before the collection \( L_{n-1,k} (p_1 - \binom{n-1}{k-1}) \) of \( k \) sets on \([n-1]\). That is, the \( p_2 \) \( k \)-sets form the collection \( P_{n-1,k} \left( \binom{p_1 - \binom{n-1}{k-1}}{k-1} \right) (p_2) \). Thus

\[
|\triangle_N L_{n,k} (p_1 + p_2) | = |\triangle_N L_{n,k} (p_1) | + |\triangle_N P_{n-1,k} \left( \binom{p_1 - \binom{n-1}{k-1}}{k-1} \right) (p_2) |
\]

(by Observation 2.11)

\[
\geq |\triangle_N L_{n,k} (p_1) | + |\triangle_N L_{n-1,k} (p_2) |
\]

(by Note 2.52 and Theorem 2.47)

as required. \( \square \)

**Example 7.7.** Let \( n = 5, k = 3, p_1 = 7, p_2 = 2 \). Then \( |\triangle_N L_{5,3} (7 + 2) | = |\triangle_N \{124, 134, 234, 125, 135, 235, 145, 245, 345\}| = |\triangle_N L_{5,3} (7) | + |\triangle_N \{124, 134\}| = \)
Lemma 7.8. If $\binom{n-2}{k-2} \leq p_1 \leq \binom{n-1}{k-1}$ and $p_2 \leq \binom{n-2}{k-1}$, then

$$|\Delta_N L_{n-1} (p_1 + p_2)| \geq |\Delta_N L_{n-2,k-1} (p_1)| + |\Delta_N L_{n-2,k-1} (p_2)|.$$  

Proof. Let $B$ denote the $p_2$ consecutive $k$-sets which come before the last $p_1$ $k$-sets in squashed order. We consider two cases.

(i) Assume that $p_1 + p_2 \geq \binom{n-1}{k-1}$. It can be assumed that $B$ consists of the last $q_1$ $k$-subsets of $[n-1]$ and the first $q_2$ $k$-subsets having $n$ as an element. Given the bounds on $p_1$ and $p_2$, and so on $q_1$ and $q_2$, $B$ is the collection $(L_{n-2,k-1}(q_1) \uplus \{n-1\}) \cup (F_{n-2,k-1}(q_2) \uplus \{n\})$. Then

$$|\Delta_N B| = |\Delta_N L_{n-2,k-1} (q_1)| + |\Delta_N F_{n-2,k-1} (q_2)|$$

(by Observation 2.10

and Lemma 2.30)

$$\geq |\Delta_N L_{n-2,k-1} (p_2)|.$$

(by Observation 2.9

and Corollary 2.55)  

(ii) Assume that $p_1 + p_2 < \binom{n-1}{k-1}$. Then $B$ is a collection of $p_2$ consecutive $(k+1)$-subsets of $[n]$ all containing $n$ as an element. Thus $B$ corresponds to a collection $C_{n-1,k-1}(p_2)$ by Lemma 2.30. Given that $B$ comes immediately before $L_{n,k}(p_1)$ with $p_1 \geq \binom{n-2}{k-2}$, no set in $B$ contains the element $n-1$. It follows that $B$ corresponds to a collection $C_{n-2,k-1}(p_2)$. Then

$$|\Delta_N B| = |\Delta_N C_{n-2,k-1} (p_2)|$$

$$\geq |\Delta_N L_{n-2,k-1} (p_2)|.$$

(by Theorem 2.47)
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We conclude that

\[ |\Delta N L_{n,k} (p_1 + p_2)| = |\Delta N L_{n,k} (p_1)| + |\Delta N P_{n,k}^n (p_2)| \]

(by Observation 2.11)

\[ = |\Delta N L_{n-1,k-1} (p_1)| + |\Delta N B| \]

(by Lemma 2.31 as \( p_1 \leq \binom{n-1}{k-1} \))

\[ \geq |\Delta N L_{n-1,k-1} (p_1)| + |\Delta N L_{n-2,k-1} (p_2)| \]

(by (7.1) and (7.2))
as required.

Example 7.9. Let \( n = 6 \), \( k = 4 \), \( p_1 = 7 \) and \( p_2 = 4 \). Then \( L_{6,4}(p_1+p_2) = L_{6,4}(7+4) = \{2345, 1236, 1246, 1346, 2346, 1256, 1356, 2356, 1456, 2456, 3456 \} \). If \( B \) is as defined in the proof of Lemma 7.8, then \( B = \{2345, 1236, 1246, 1346\} \). Thus \( |\Delta N B| \geq |\Delta N \{1236, 1246, 1346, 2346\}| = |\Delta N (L_{4,3}(4) \uplus \{6\})| = |\Delta N L_{4,3}(4)| \) so that

\[ |\Delta N L_{6,4}(7+4)| \geq |\Delta N L_{5,3}(7)| + |\Delta N L_{4,3}(4)|. \]

\( \square \)

Lemma 7.10. If \( p_1, p_2 \leq \binom{n-2}{k-1} \) and \( p_1 + p_2 \geq \binom{n-2}{k-1} \), then

\[ \left| \Delta N L_{n,k} \left( \binom{n-2}{k-2} + p_1 + p_2 \right) \right| \]

\[ \geq \left| \Delta N L_{n-2,k-2} \left( \binom{n-2}{k-2} \right) \right| + \left| \Delta N L_{n-2,k-1} (p_1) \right| + \left| \Delta N L_{n-2,k-1} (p_2) \right|. \]

Proof. Note that \( \binom{n-2}{k-2} + p_1 \leq \binom{n-1}{k-1} \). Let \( B \) denote the \( p_2 \) consecutive \( k \)-sets that come before the last \( \binom{n-2}{k-2} + p_1 \) \( k \)-sets in squashed order. The same argument as the one developed in (i) of the proof of Lemma 7.8 shows that

\[ B = (L_{n-2,k-1}(q_1) \uplus \{n-1\}) \cup (F_{n-2,k-1}(q_2) \uplus \{n\}) \]

\( (7.3) \)

for some \( q_1, q_2 \in N \).
Then,

$$\left| \triangle_N L_{n,k} \left( \binom{n-2}{k-2} + p_1 + p_2 \right) \right| = \left| \triangle_N L_{n,k} \left( \binom{n-2}{k-2} + p_1 \right) \right| + \left| \triangle_N P_{n,k}^{(\binom{n-2}{k-2} + p_1)} [p_2] \right|$$

(by Observation 2.11)

$$= \left| \triangle_N L_{n-1,k-1} \left( \binom{n-2}{k-2} + p_1 \right) \right| + |\triangle_N \mathcal{B}|$$

(by Lemma 2.31 as $\binom{n-2}{k-2} + p_1 \leq \binom{n-1}{k-1}$)

$$\geq \left| \triangle_N L_{n-1,k-1} \left( \binom{n-2}{k-2} \right) \right| + |\triangle_N L_{n-2,k-1} (p_1)| + |\triangle_N \mathcal{B}|$$

(by Lemma 7.4)

$$= \left| \triangle_N L_{n-2,k-2} \left( \binom{n-2}{k-2} \right) \right| + |\triangle_N L_{n-2,k-1} (p_1)| + |\triangle_N \mathcal{B}|$$

(by Lemma 2.31)

$$= \left| \triangle_N L_{n-2,k-2} \left( \binom{n-2}{k-2} \right) \right| + |\triangle_N L_{n-2,k-1} (p_1)|$$

+ $|\triangle_N L_{n-2,k-1} (q_1)| + |\triangle_N \mathcal{B}|$

(by (7.3), Observation 2.10 and Lemma 2.30)

$$\geq \left| \triangle_N L_{n-2,k-2} \left( \binom{n-2}{k-2} \right) \right| + |\triangle_N L_{n-2,k-1} (p_1)| + |\triangle_N L_{n-2,k-1} (p_2)|$$

(by Observation 2.9 and Corollary 2.55)

as required. \qed

**Example 7.11.** Let $n = 6$, $k = 4$, $p_1 = 2$ and $p_2 = 3$. Then $L_{6,4} \left( \binom{4}{2} + p_1 + p_2 \right) = L_{6,4} \left( \binom{4}{2} + 2 + 3 \right) = \{2345, 1236, 1246, 1346, 2346, 1256, 1356, 2356, 1456, 2456, 3456\}$. If $\mathcal{B}$ is as defined in the proof of Lemma 7.10, then $\mathcal{B} = \{2345, 1236, 1246\}$. Thus $|\triangle_N \mathcal{B}| \geq |\triangle_N \{1246, 1346, 2346\}| = |\triangle_N (L_{4,3}(3) \cup \{6\})| = |\triangle_N L_{4,3}(3)|$ so that $|\triangle_N L_{6,4} \left( \binom{4}{2} + 2 + 3 \right)| \geq |\triangle_N L_{4,2} \left( \binom{4}{2} \right)| + |\triangle_N L_{4,3}(2)| + |\triangle_N L_{4,3}(3)|$. \hfill \Box

Lemma 7.12 below is the last lemma in this section. It concerns appropriately chosen collections of $\binom{n-2}{k-1}$ sets.

**Lemma 7.12.** If $k$ and $n \in \mathbb{Z}^+$ are such that $\frac{k+1}{n-k} + \frac{n-k-1}{k} \geq 2$, then

$$\left| \triangle_N L_{n,k+1} \left( \binom{n-2}{k-1} \right) \right| + \left| \nabla_N F_{n,k-1} \left( \binom{n-2}{k-1} \right) \right| \geq 2 \binom{n-2}{k-1}.$$
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Proof. Assume that \( k \) and \( n \in \mathbb{Z}^+ \) are such that \( \frac{k+1}{n-k} + \frac{n-k-1}{k} \geq 2 \). Then it is easily checked that

\[
\binom{n-2}{k-2} + \binom{n-2}{k} \geq 2 \binom{n-2}{k-1}.
\] (7.4)

Now,

\[
\begin{align*}
\triangle_n \mathcal{L}_{n,k+1} \left( \binom{n-2}{k-1} \right) &+ \triangle_n \mathcal{F}_{n,k-1} \left( \binom{n-2}{k-1} \right) \\
&= \triangle_n \mathcal{L}_{n-2,k-1} \left( \binom{n-2}{k-1} \right) + \triangle_n \mathcal{F}_{n-2,k-1} \left( \binom{n-2}{k-1} \right) \\
&= \left( \binom{n-2}{k-2} + \binom{n-2}{k} \right) \\
&\geq 2 \binom{n-2}{k-1}.
\end{align*}
\]

(by Lemmas 2.31 and 2.28)

(by Observations 2.7)

(by Observations 2.9 and 2.8)

(by (7.4))

This proves the lemma. \( \square \)

7.3 Proof Outline of Theorem 7.1.(i): Figure 7.1

Before presenting the proof of Theorem 7.1.(i), we give its general outline in Figure 7.1. The proof is long and is made up of a sequence of 13 propositions. Some lemmas involving algebraic inequalities are used in the course of the proof. These lemmas are labelled Lemma E.i, \( i \in \mathbb{Z}^+ \), and are included in Appendix E.

The theorem is easy to prove for \( 1 \leq k \leq \frac{n+1}{3} \) and this is done first. A computer check shows that Theorem 7.1.(i) holds for values of \( n \leq 32 \). The rest of the proof is an induction on \( n \) and considers the two cases \( \frac{n+1}{3} < k \leq \frac{n}{2} \) and \( k > \frac{n}{2} \). These cases are further subdivided into many cases and subcases, according to the values of
Part A

1) \(1 \leq k \leq \frac{n+1}{3}\) Proposition 7.13

2) \(n \leq 32\) Proposition 7.14

3) \(\frac{n+1}{3} < k \leq \frac{n}{2}\)

3.1) \(1 \leq p \leq \binom{n-1}{k-2}\) Proposition 7.16

3.2) \(\binom{n-1}{k-2} < p \leq \binom{n}{k-1}\)

3.2.1) \(\binom{n-1}{k-2} < \binom{n-2}{k-1}\)

3.2.1.1) \(\binom{n-1}{k-2} < p \leq \binom{n-2}{k-1}\) Proposition 7.17

3.2.1.2) \(\binom{n-2}{k-1} < p \leq \binom{n-1}{k}\) Proposition 7.18

3.2.1.3) \(\binom{n-1}{k} < p \leq \binom{n}{k-1}\) Proposition 7.19

3.2.2) \(\binom{n-1}{k-2} \geq \binom{n-2}{k-1}\)

3.2.2.1) \(\binom{n-1}{k-2} < p \leq \binom{n-1}{k-2} + \binom{n-2}{k}\) Proposition 7.20

3.2.2.2) \(\binom{n-1}{k-2} + \binom{n-2}{k} < p \leq \binom{n}{k-1}\)

3.2.2.2.1) \(\frac{k-1}{n-k} + \frac{n-k-1}{k} \geq 2\) Proposition 7.21

Part B

3.2.2.2.2) \(\frac{k-1}{n-k} + \frac{n-k-1}{k} < 2\) Proposition 7.22

Part C

4) \(k > \frac{n}{2}\)

4.1) \(1 \leq p \leq \binom{n-1}{k}\) Proposition 7.57

4.2) \(\binom{n-1}{k} < p \leq \binom{n}{k+1}\)

4.2.1) \(\binom{n-2}{k-3} > \binom{n-1}{k}\) Proposition 7.58

4.2.2) \(\binom{n-2}{k-3} \leq \binom{n-1}{k}\)

4.2.2.1) \(\binom{n-1}{k} < p \leq \binom{n-1}{k-2}\) Proposition 7.59

4.2.2.2) \(\binom{n-1}{k-2} < p \leq \binom{n}{k+1}\) Proposition 7.60

Figure 7.1: Outline of the cases considered in the proof of Theorem 7.1.(i)
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$k$ and $p$ in Theorem 7.1(i). Recall that by Note 7.2 we only consider values of $p$ in $\mathbb{Z}^+$.

The proof of Theorem 7.1(i) is presented in three parts. Part A considers most of the cases when $k \leq \frac{n}{2}$, this includes Proposition 7.14 up to and including Proposition 7.21. Part B presents Proposition 7.22 whose proof is long and which is the last case considered when $k \leq \frac{n}{2}$. Part C proves Theorem 7.1(i) in the case $k > \frac{n}{2}$.

The proposition associated with each case is shown in the proof outline. In the course of proving each proposition the numbering of each case under consideration will follow the numbering given in Figure 7.1.

7.4 The Proof of Theorem 7.1(i): Part A

1) $1 \leq k \leq \frac{n+1}{3}$

Proposition 7.13. Theorem 7.1(i) holds for $1 \leq k \leq \frac{n+1}{3}$.

Proof. By Note 7.2 only values of $p$ in $\mathbb{Z}^+$ are considered. Assume that $k = 1$, then $p = 1$. By Corollary 2.41 and Lemma 2.60, $|\Delta N\left(n,k+1\right)(p)| + |\nabla L_{n,k-1}(p)| = |\Delta N\left(n,2\right)(1)| + |\nabla L_{n,0}(1)| = 4$ for $n = 2$, and $|\Delta N\left(n,2\right)(1)| + |\nabla L_{n,0}(1)| = n$ for $n > 2$. Thus Theorem 7.1(i) holds for $k = 1$.

Now let $k > 1$. Using Sperner’s lemma together with Observation 2.9, we see that

$$|\nabla N\left(n,k-1\right)(p)| = |\nabla L_{n,k-1}(p)| \geq \frac{n-k+1}{k}p \geq 2p$$

for $k \leq \frac{n+1}{3}$.

We consider two cases.

(i) $p \neq \left(\begin{array}{c}n \\ k-1\end{array}\right)$. Then Sperner’s lemma is a strict inequality by Lemma 2.35, and $|\nabla L_{n,k-1}(p)| > \frac{n-k+1}{k}p \geq 2p$.

(ii) $p = \left(\begin{array}{c}n \\ k-1\end{array}\right)$. Then $|\nabla L_{n,k-1}(p)| = \frac{n-k+1}{k}p \geq 2p$ by Lemma 2.35. We show that
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\[ |\Delta_N L_{n,k+1}(p)| \neq 0. \] If \( p = \binom{n}{k-1} \) then \( p > \binom{n-2}{k-1} \), so that

\[
|\Delta_N L_{n,k+1}(p)| \geq \left| \Delta_N L_{n,k+1}\left(\binom{n-2}{k-1}\right)\right|
\]

\[
= \left| \Delta_N L_{n-2,k-1}\left(\binom{n-2}{k-1}\right)\right|
\]

(by Lemma 2.31)

\[
= \left| \Delta_N F_{n-2,k-1}\left(\binom{n-2}{k-1}\right)\right|
\]

(by Observation 2.7)

\[
= \left| \Delta F_{n-2,k-1}\left(\binom{n-2}{k-1}\right)\right|
\]

(by Observation 2.9)

\[
= \left(\frac{n-2}{k-2}\right)
\]

(by Observation 2.8)

\[
> 0.
\]

(as \( k > 1 \))

Hence \( |\Delta_N L_{n,k+1}(p)| + |\nabla L_{n,k-1}(p)| > 2p. \)

\[ \square \]

2) \( n \leq 32 \)

Proposition 7.14. Theorem 7.1.(i) holds for \( n \leq 32 \).

Proof. Exhaustive computations show that Theorem 7.1.(i) holds for \( n \leq 32 \). This is done using Algorithm D.1 which, together with the proof of its correctness, its implementation, and its output for \( n \leq 32 \) is included in Appendix D. The implementation of Algorithm D.1 (see Page 217) computes the minimum value of the ratio \( \frac{\nabla L_{n,k-1}(p) + |\Delta_N L_{n,k+1}(p)|}{p} \), where the minimum is taken, for \( n \) and \( k \) fixed, over all values of \( p \) such that \( 1 \leq p \leq \min \left\{ \binom{n}{k+1}, \binom{n}{k-1} \right\} \).

These minimum values are shown in the second column of the output beginning Page 221 in the form \( \min = \) and are computed for all \( n \) and \( k \) such that \( n \leq 32 \) and \( \left\lfloor \frac{n+1}{3} \right\rfloor \leq k < n - 1 \). Theorem 7.1.(i) holds for \( k \leq \frac{n+1}{3} \) by Proposition 7.13. That
Theorem 7.1.(i) holds when \( k = n - 1 \) is shown in Appendix D (see Lemma D.9.) The fact that these values always exceed 2 provides verification of Theorem 7.1.(i) for \( n \leq 32 \). \( \square \)

The proofs of all subsequent propositions in this section and Sections 7.5 and 7.6 are proofs by induction. In each proposition the induction hypothesis is

**Induction Hypothesis 7.15.** *Theorem 7.1 holds for all positive integers less than \( n \).*

By Proposition 7.14, Theorem 7.1.(i) holds for any positive integer less than 33 and so Theorem 7.1 holds for any positive integer less than 33 by Lemma 7.3. Induction Hypothesis 7.15 is implicitly assumed to hold when proving Propositions 7.16 to 7.60. Each of the Propositions 7.16 to 7.59 considers particular values of \( p \) and \( k \) and shows, assuming that Induction Hypothesis 7.15 holds, that Theorem 7.1.(i) holds for \( n \) for those values of \( p \) and \( k \). Put together, Propositions 7.16 to 7.59 prove that Theorem 7.1.(i) holds for \( n \) and for all values of \( p \), \( 0 \leq p \leq \min \left( \binom{n}{k+1}, \binom{n}{k-1} \right) \), with \( 1 \leq k < n \).

In each proposition we must show that the collections \( L_{n,k+1}(p) \) and \( L_{n,k-1}(p) \) satisfy Theorem 7.1.(i). That is, we show that \( |\triangle_N L_{n,k+1}(p)| + |\nabla N L_{n,k-1}(p)| > 2p \) for \( p > 0 \). This will be achieved in the following manner. The collections \( L_{n,k+1}(p) \) and \( L_{n,k-1}(p) \) are partitioned into \( \{A_1, A_2, \ldots, A_m\} \) and \( \{B_1, B_2, \ldots, B_m\} \) respectively with \( |A_i| = |B_i| \) for \( i = 1, \ldots, m \). It is shown that for each \( i, i = 1, \ldots, m \), \( |\triangle_N A_i| + |\nabla N B_i| \geq 2|A_i| = 2|B_i| \) with a strict inequality occurring for at least one value of \( i \). Showing that this holds is achieved by using two different methods. In order to simplify the present discussion, assume for now that \( A_i \) and \( B_i \neq \emptyset \) for \( i = 1, \ldots, m \).

The first method involves showing that \( |\triangle_N A_i| + |\nabla N B_i| \geq 2|A_i| \) by applying various results such as Sperner's lemma or Lemma 7.12. Sometimes the argument applied is one of Propositions 7.16 to 7.59, which will have been proved beforehand. In this latter case it is actually shown that \( |\triangle_N A_i| + |\nabla N B_i| > 2|A_i| \).

The second method involves finding correspondences between the collections \( A_i \) and
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$B_i$ on $[n]$ and collections of subsets of $[n-j]$, $0 < j < n$. Using these correspondences it is then shown that $|\triangle_N A_i| + |\nabla_N B_i| > 2|A_i|$ by applying Induction Hypothesis 7.15. Observation 2.10 is then applied to complete the proof that Theorem 7.1.(i) holds for the value of $p$ considered. One must note that the process of finding appropriate partitions for the collections $L_{n,k+1}(p)$ and $L_{n,k-1}(p)$ is relatively easy except in the case dealt with in Part B of the proof (Proposition 7.22). There the complexity of the partition process accounts for the length of the proof of Proposition 7.22.

We first consider values of $k$ such that $\frac{n+1}{3} < k \leq \frac{n}{2}$.

3) $\frac{n+1}{3} < k \leq \frac{n}{2}$

The first values of $p$ considered for $\frac{n+1}{3} < k \leq \frac{n}{2}$ are those such that $1 \leq p \leq \binom{n-1}{k-2}$.

3.1) $1 \leq p \leq \binom{n-1}{k-2}$

**Proposition 7.16.** Let $n > 32$ and $\frac{n+1}{3} < k \leq \frac{n}{2}$. Then Theorem 7.1.(i) holds for $p \leq \binom{n-1}{k-2}$.

**Proof.** For $k \leq \frac{n}{2}$, $\binom{n-1}{k-2} < \binom{n-1}{k}$. As $p \leq \binom{n-1}{k-2}$ by assumption this implies that $p < \binom{n-1}{k}$. Hence

$$\left|\triangle_N L_{n,k+1}(p)\right| + \left|\nabla_N L_{n,k-1}(p)\right|$$

$$= \left|\triangle_N L_{n-1,k}(p)\right| + \left|\nabla_N L_{n-1,k-2}(p)\right|$$

(by Lemma 2.31)

as $p < \binom{n-1}{k}$ and $p \leq \binom{n-1}{k-2}$

$$> 2p.$$  

(by Induction Hypothesis 7.15 as $p > 0$)

Figure 7.2 illustrates the collections $L_{n,k+1}(p)$ and $L_{n,k-1}(p)$ in the proof of Proposition 7.16.
There are two more cases to consider for \( n - 1 < k \leq \frac{n}{2} \), depending on whether \( \binom{n-1}{k-2} \)
is smaller or larger than \( \binom{n-2}{k-1} \). When \( \binom{n-1}{k-2} < \binom{n-2}{k-1} \) three ranges of values of \( p \) are considered.

3.2.1) \( \binom{n-1}{k-2} < \binom{n-2}{k-1} \)

3.2.1.1) \( \binom{n-1}{k-2} < p \leq \binom{n-2}{k-1} \)

**Proposition 7.17.** Let \( n > 32 \) and \( \frac{n+1}{4} < k \leq \frac{n}{2} \) with \( \binom{n-1}{k-2} < \binom{n-2}{k-1} \). Then Theorem 7.1.(i) holds for \( \binom{n-1}{k-2} < p \leq \binom{n-2}{k-1} \).

Before proceeding to the proof of Proposition 7.17 we present Figure 7.3 which is an illustration of the partitioning of the collections \( L_{n,k+1}(p) \) and \( L_{n,k-1}(p) \) used in the proof. We also outline the approach used in the proof. A similar approach is taken in the proofs of the subsequent propositions.

Figure 7.3 depicts a decomposition of the \( (k+1) \)-sets which is relevant to Proposition 7.17. Note that the collection \( L_{n,k+1}(\binom{n-1}{k}) \) consists of the \( \binom{n-2}{k} (k+1) \)-sets having \( n \) but not \( n-1 \) as an element, which are followed in squashed order by the
that the first \( \binom{n-2}{k-1} \) \((k+1)\)-sets having \( n-1 \) and \( n \) as elements. Note also that the collection 
\[ L_{n,k+1}(\binom{n-2}{k+1}) \]
consists of the \( \binom{n-2}{k-1} \) \((k+1)\)-sets having \( n-1 \) and \( n \) but not \( n-2 \) as elements, which are followed in squashed order by the \( \binom{n-3}{k-1} \) \((k+1)\)-sets having \( n-2 \), \( n-1 \) and \( n \) as elements.

For \( p' = p - \binom{n-1}{k-2} \) it will be shown that the first \( p' \) sets in \( L_{n,k-1}(p) \) correspond to 
\( L_{n-1,k-1}(p') \) and that the new-shadow of the first \( p' \) sets in \( L_{n,k+1}(p) \) is no smaller than the new-shadow of \( L_{n-1,k+1}(p') \). This is done as follows. It is easy to see that the first \( p' \) sets in \( L_{n,k-1}(p) \) correspond to \( L_{n-1,k-1}(p') \). Further, note that the first \( p' \) sets in \( L_{n,k+1}(p) \) correspond to some collection \( C_{n-3,k-1}(p') \) and recall that 
\[ |\Delta N C_{n-3,k-1}(p')| \geq |\Delta N L_{n-3,k-1}(p')| \]
by Theorem 2.47. It is left to show that \( L_{n-3,k-1}(p') \) corresponds to \( L_{n-1,k+1}(p') \). This is done by noting that \( L_{n-3,k-1}(p') \) consists of the last \( p' \) \((k+1)\)-sets having \( n-1 \) and \( n \) but not \( n-2 \) as elements (as \( p' \leq \binom{n-3}{k-1} \)) and that \( L_{n-1,k+1}(p') \) consists of the last \( p' \) \((k+1)\)-sets having \( n-2 \) and \( n-1 \) but not \( n \) as elements.

Thus \( L_{n-3,k-1}(p') \) corresponds to \( L_{n-1,k+1}(p') \) by Lemma 2.30 and 
\[ |\Delta N F'(p', L_{n,k+1}(p))| \geq |\Delta N L_{n-1,k+1}(p')| \]
Proposition 7.17 then follows from Induction Hypothesis 7.15. We now prove Proposition 7.17 formally.

**Proof of Proposition 7.17.** Recall that \( \binom{n-1}{k-2} < p \leq \binom{n-1}{k-1} \) and let \( p = p' + \binom{n-1}{k-2} \). Then \( p' > 0 \). We have 
\[
|\Delta N L_{n,k+1}(p)| = |\Delta N L_{n-2,k-1}(p)|
\]
(by Lemma 2.31 as \( p \leq \binom{n-3}{k-1} \))
\[
= |\Delta N L_{n-2,k-1}(\binom{n-1}{k-2} + p')|
\]
\[
\geq |\Delta N L_{n-2,k-1}(\binom{n-1}{k-2})| + |\Delta N L_{n-3,k-1}(p')|
\]
(by Lemma 7.6 as \( \binom{n-1}{k-2} > \binom{n-3}{k-1} \))

so that
\[
|\Delta N L_{n,k+1}(p)| \geq |\Delta N L_{n-1,k}(\binom{n-1}{k-2})| + |\Delta N L_{n-1,k+1}(p')|.
\]
(by Lemma 2.31)
Also,

\[
\left| \nabla N^2 L_{n,k-1}(p) \right| = \left| \nabla N^2 L_{n,k-1} \left( \binom{n}{k} + p' \right) \right|
\]

\[
= \left| \nabla N^2 L_{n-1,k-2} \left( \binom{n-1}{k-2} \right) \right| + \left| \nabla N^2 L_{n-1,k-1}(p') \right|.
\]

(by Lemma 7.4)

Thus

\[
\left| \nabla N^2 L_{n,k+1}(p) \right| + \left| \nabla N^2 L_{n,k-1}(p) \right|
\]

\[
\geq \left| \nabla N^2 L_{n-1,k-1} \left( \binom{n-1}{k-2} \right) \right| + \left| \nabla N^2 L_{n-1,k+1}(p') \right|
\]

\[
+ \left| \nabla N^2 L_{n-1,k-2} \left( \binom{n-1}{k-2} \right) \right| + \left| \nabla N^2 L_{n-1,k-1}(p') \right|
\]

(by (7.5) and (7.6))

\[
> 2 \binom{n-1}{k-2} + 2p'
\]

(by Induction Hypothesis 7.15

as \( \binom{n-1}{k-2}, p' > 0 \))

\[
= 2p.
\]

\[\square\]

The next values of \( p \) considered when \( \binom{n-1}{k-2} < \binom{n-2}{k-1} \) are such that \( \binom{n-2}{k-1} < p \leq \binom{n-1}{k-1} \).

3.2.1.2) \( \binom{n-2}{k-3} < p \leq \binom{n-1}{k} \)

**Proposition 7.18.** Let \( n > 32 \) and \( \frac{n+1}{3} < k \leq \frac{n}{2} \) with \( \binom{n-1}{k-2} < \binom{n-2}{k-1} \). Then Theorem 7.1(i) holds for \( \binom{n-2}{k-1} < p \leq \binom{n-1}{k} \).

**Proof.** The partitioning of \( L_{n,k+1}(p) \) and \( L_{n,k-1}(p) \) used in the proof is illustrated by Figure 7.4.

We begin by proving the following claim.
Figure 7.4: Partitioning of $L_{n,k+1}(p)$ and $L_{n,k-1}(p)$ in the proof of Proposition 7.18

Claim.

$$\left| \Delta_N L_{n-2,k-1} \left( \binom{n-2}{k-1} \right) \right| + \left| \nabla_N L_{n-1,k-2} \left( \binom{n-1}{k-2} \right) \right| \geq 2 \binom{n-2}{k-1}. \quad (7.7)$$

Proof of the claim. We have

$$\left| \Delta_N L_{n-2,k-1} \left( \binom{n-2}{k-1} \right) \right| + \left| \nabla_N L_{n-1,k-2} \left( \binom{n-1}{k-2} \right) \right|$$

$$= \left| \Delta_N F_{n-2,k-1} \left( \binom{n-2}{k-1} \right) \right| + \left| \nabla_N L_{n-1,k-2} \left( \binom{n-1}{k-2} \right) \right|$$

(by Observation 2.7)

$$= \binom{n-2}{k-2} + \binom{n-1}{k-1}$$

(by Observations 2.9 and 2.8)

$$\geq 2 \binom{n-2}{k-1}$$

since for $k > \frac{n+1}{3}$, $\frac{n+1}{3} + \frac{n+1}{n-1} \geq 2$. This proves the claim.

Let $p = p' + \binom{n-2}{k-1}$ and $p = p'' + p' + \binom{n-1}{k-2}$. As $p > \binom{n-2}{k-1}$, $p' > 0$. Also $p'' = p - p' - \binom{n-1}{k-2} = \binom{n-2}{k-2} - \binom{n-1}{k-2}$. As by assumption $\binom{n-2}{k-1} > \binom{n-1}{k-2}$ it follows that
$\gamma'' > 0$. Then

$$|
abla \Delta \! L_{n,k+1} (p)| = \bigg| \Delta \Delta \! L_{n,k+1} \left( \begin{array}{c} n - 2 \\ k - 1 \end{array} \right) + p' \bigg|$$

$$= \bigg| \Delta \Delta \! L_{n-1,k} \left( \begin{array}{c} n - 2 \\ k - 1 \end{array} \right) + p' \bigg|$$

(by Lemma 2.31)

$$= \bigg| \Delta \Delta \! L_{n-2,k-1} \left( \begin{array}{c} n - 2 \\ k - 1 \end{array} \right) + \big| \nabla \Delta \! L_{n-2,k} (p') \big| \bigg|$$

(by Lemma 7.4)

so that

$$|
abla \Delta \! L_{n,k+1} (p)| = \bigg| \nabla \Delta \! L_{n-2,k-1} \left( \begin{array}{c} n - 2 \\ k - 1 \end{array} \right) \bigg| + \big| \nabla \Delta \! L_{n-1,k+1} (p') \big|.$$

(7.8)

(by Lemma 2.31)

For the new-shade of $L_{n,k-1} (p)$ we write:

$$|
abla \nabla \! L_{n,k-1} (p)| = \bigg| \nabla \nabla \! L_{n,k-1} \left( \begin{array}{c} n - 1 \\ k - 2 \end{array} \right) + p' + p'' \bigg|$$

$$= \bigg| \nabla \nabla \! L_{n-1,k-2} \left( \begin{array}{c} n - 1 \\ k - 2 \end{array} \right) \bigg| + \big| \nabla \nabla \! L_{n-2,k-1} (p') + p'' \big|$$

(by Lemma 7.4)

$$= \bigg| \nabla \nabla \! L_{n-1,k-2} \left( \begin{array}{c} n - 1 \\ k - 2 \end{array} \right) \bigg|$$

$$+ \big| \nabla \nabla \! L_{n-2,k-1} (p') \big| + \big| \nabla \nabla \! P_{n-1,k-1} (p'') \big|$$

(by Observations 2.9 and 2.11)

so that

$$|
abla \nabla \! L_{n,k-1} (p)| = \bigg| \nabla \nabla \! L_{n-2,k-2} \left( \begin{array}{c} n - 1 \\ k - 2 \end{array} \right) \bigg|$$

$$+ \big| \nabla \nabla \! L_{n-1,k-1} (p') \big| + \big| \nabla \nabla \! P_{n-1,k-1} (p'') \big|.$$

(7.9)
We conclude that
\[
|\triangle_{N} L_{n,k+1}(p)| + |\nabla_{N} L_{n,k-1}(p)| \\
= |\triangle_{N} L_{n-2,k-1}\left(\binom{n-2}{k-1}\right)| + |\triangle_{N} L_{n-1,k+1}(p')| \\
+ |\nabla_{N} L_{n-1,k-2}\left(\binom{n-1}{k-2}\right)| + |\nabla_{N} L_{n-1,k-1}(p')| + |\nabla_{N} P_{n-1,k-1}(p'')| \\
(\text{by (7.8) and (7.9))}
\]
\[
\geq |\triangle_{N} L_{n-2,k-1}\left(\binom{n-2}{k-1}\right)| + |\triangle_{N} L_{n-1,k+1}(p')| \\
+ |\nabla_{N} L_{n-1,k-2}\left(\binom{n-1}{k-2}\right)| + |\nabla_{N} L_{n-1,k-1}(p')| \\
> 2\binom{n-2}{k-1} + 2p' \\
(\text{by (7.7)}
\]
and Induction Hypothesis 7.15 as \( p' > 0 \)
\[
= 2p.
\]
\[
\]

The next values of \( p \) to consider are the last values which need to be discussed for the case when \( \binom{n-1}{k-2} < \binom{n-2}{k-1} \). Note that for \( k \leq \frac{n}{2} \), \( \min \left\{ \binom{n}{k+1}, \binom{n}{k-1} \right\} = \binom{n}{k-1} \).

3.2.1.3) \( \binom{n-1}{k} < p \leq \binom{n}{k-1} \)

**Proposition 7.19.** Let \( n > 32 \) and \( \frac{n+1}{3} < k \leq \frac{n}{2} \) with \( \binom{n-1}{k-2} < \binom{n-2}{k-1} \). Then Theorem 7.1.(i) holds for \( \binom{n-1}{k} < p \leq \binom{n}{k-1} \).

**Proof.** The partitioning of \( L_{n,k+1}(p) \) and \( L_{n,k-1}(p) \) used in the proof is illustrated by Figure 7.5.

By Lemma E.1, if \( \binom{n-1}{k-2} < \binom{n-2}{k-1} \) then \( \frac{k+1}{n-k} + \frac{n-k-1}{k} \geq 2 \) for \( n > 32 \). Thus Lemma 7.12 can be applied.
Figure 7.5: Partitioning of $L_{n,k+1}(p)$ and $L_{n,k-1}(p)$ in the proof of Proposition 7.19

Let $p = \binom{n-2}{k-1} + p' + \binom{n-1}{k-2}$. Then

\[
p' = p - \binom{n-2}{k-1} - \binom{n-1}{k-2} > \binom{n-1}{k} - \binom{n-2}{k-1} - \binom{n-1}{k-2}, \tag{as $p > \binom{n-1}{k}$ by assumption)}
\]

\[
= \binom{n-2}{k} + \binom{n-2}{k-1} - \binom{n-2}{k-2} - \binom{n-1}{k-1} - \binom{n-1}{k-2} = \binom{n-2}{k} - \binom{n-1}{k-2} > 0.
\]

(by Lemma E.2 as $\binom{n-1}{k-2} < \binom{n-2}{k-1}$)

Also,

\[
p' = p - \binom{n-2}{k-1} - \binom{n-1}{k-2} \leq \binom{n}{k-1} - \binom{n-2}{k-1} - \binom{n-1}{k-2} \cdot \tag{as $p \leq \binom{n}{k-1}$ by assumption)}
That is,
\[
p' \leq \binom{n-1}{k-1} - \binom{n-2}{k-1} = \binom{n-2}{k-2} \leq \binom{n-2}{k}
\]
(as \(\binom{n-2}{k-1} \leq \binom{n-2}{k}\) for \(k \leq \frac{n}{2}\))
so that
\[
p' \leq \binom{n-2}{k-2} \quad \text{and} \quad p' \leq \binom{n-2}{k}.
\]
Note that \(p - \binom{n-2}{k-1} = p' + \binom{n-1}{k-2} > \binom{n-2}{k-1} = \binom{n-2}{k}\). Also, recall that \(\binom{n-1}{k-2} < \binom{n-2}{k}\) by assumption. Then,
\[
|\Delta_N L_{n,k+1}(p)| = |\Delta_N L_{n,k+1}\left(\binom{n-2}{k-1} + \binom{n-1}{k-2} + p'\right)| \\
\geq |\Delta_N L_{n-2,k-1}\left(\binom{n-2}{k-1}\right)| + |\Delta_N L_{n-2,k}\left(\binom{n-1}{k-2}\right)| + |\Delta_N L_{n-2,k}(p')| \\
\geq |\Delta_N L_{n-2,k-1}\left(\binom{n-2}{k-1}\right)| + |\Delta_N L_{n-2,k-1}\left(\binom{n-1}{k-2}\right)| + |\Delta_N L_{n-2,k}(p')| \\
\quad \text{(by Lemma 7.10 as } p' + \binom{n-1}{k-2} > \binom{n-2}{k} \text{ and } \binom{n-1}{k-2}, p' \leq \binom{n-2}{k}) \\
\text{(by Theorem 2.48)}
\]
so that
\[
|\Delta_N L_{n,k+1}(p)| \\
\geq |\Delta_N L_{n,k+1}\left(\binom{n-2}{k-1}\right)| + |\Delta_N L_{n-1,k}\left(\binom{n-1}{k-2}\right)| + |\Delta_N L_{n-2,k}(p')|. \\
(7.10)
\]
(by Lemma 2.31)
For the new-shade of $L_{n,k-1}(p)$ we write

$$|\nabla_N L_{n,k-1}(p)| = |\nabla_N L_{n,k-1} \left( \left( \begin{array}{c} n-1 \\ k-2 \end{array} \right) + p' + \left( \begin{array}{c} n-2 \\ k-1 \end{array} \right) \right)|$$

$$= |\nabla_N L_{n-1,k-2} \left( \left( \begin{array}{c} n-1 \\ k-2 \end{array} \right) \right) + \nabla_N L_{n-1,k-1} \left( p' + \left( \begin{array}{c} n-2 \\ k-1 \end{array} \right) \right)\rangle$$

(by Lemma 7.4)

$$\geq |\nabla_N L_{n-1,k-2} \left( \left( \begin{array}{c} n-1 \\ k-2 \end{array} \right) \right) + \nabla_N L_{n-1,k-1} \left( p' \right) + \nabla_N F_{n-1,k-1} \left( \left( \begin{array}{c} n-2 \\ k-1 \end{array} \right) \right)\rangle$$

(by Observations 2.9 and 2.11)

$$= |\nabla_N L_{n-1,k-2} \left( \left( \begin{array}{c} n-1 \\ k-2 \end{array} \right) \right) + \nabla_N L_{n-1,k-1} \left( p' \right) + \nabla_N F_{n-1,k-1} \left( \left( \begin{array}{c} n-2 \\ k-1 \end{array} \right) \right)\rangle$$

(by Note 2.52 and Corollary 2.49)

so that

$$|\nabla_N L_{n,k-1}(p)| \geq |\nabla_N L_{n-1,k-2} \left( \left( \begin{array}{c} n-1 \\ k-2 \end{array} \right) \right) + \nabla_N L_{n-2,k-2} \left( p' \right) + \nabla_N F_{n-1,k-1} \left( \left( \begin{array}{c} n-2 \\ k-1 \end{array} \right) \right)\rangle.$$\tag{7.11}

(by Lemma 2.31 as $p' \leq \left( \begin{array}{c} n-2 \\ k-2 \end{array} \right)$)

Then

$$|\Delta_N L_{n,k+1}(p)| + |\nabla_N L_{n,k-1}(p)|$$

$$\geq |\Delta_N L_{n,k-1} \left( \left( \begin{array}{c} n-2 \\ k-1 \end{array} \right) \right) + \Delta_N L_{n-1,k} \left( \left( \begin{array}{c} n-1 \\ k-2 \end{array} \right) \right) + |\Delta_N L_{n-2,k} \left( p' \right) |$$

$$+ |\nabla_N L_{n-1,k-2} \left( \left( \begin{array}{c} n-1 \\ k-2 \end{array} \right) \right) + |\nabla_N L_{n-2,k-2} \left( p' \right) |$$

$$+ |\nabla_N F_{n-1,k-1} \left( \left( \begin{array}{c} n-2 \\ k-1 \end{array} \right) \right)\rangle.$$\tag{by (7.10) and (7.11))
It follows that
\[
|\triangle N L_{n,k+1}(p)| + |\nabla N L_{n,k-1}(p)| \\
> 2 \binom{n-2}{k-1} + 2 \binom{n-1}{k-2} + 2p'
\]
(by Lemma 7.12 and Induction Hypothesis 7.15 as \(\binom{n-1}{k-2}, p' > 0\))
\[
= 2p.
\]

This concluded the discussion for the case \(\binom{n-1}{k-2} < \binom{n-2}{k-1}\) and \(n+1 < k \leq n/2\). We now consider the case when \(\binom{n-1}{k-2} \geq \binom{n-2}{k-1}\) for \(n+1 < k \leq n/2\).

**3.2.2** \(\binom{n-1}{k-2} \geq \binom{n-2}{k-1}\)

**3.2.2.1** \(\binom{n-1}{k-2} < p \leq \binom{n-1}{k-2} + \binom{n-2}{k}\)

**Proposition 7.20.** Let \(n > 32\) and \(n+1 < k \leq n/2\) with \(\binom{n-1}{k-2} \geq \binom{n-2}{k-1}\). Then Theorem 7.1.(i) holds for \(\binom{n-1}{k-2} < p \leq \binom{n-1}{k-2} + \binom{n-2}{k}\).

**Proof.** The partitioning of \(L_{n,k+1}(p)\) and \(L_{n,k-1}(p)\) used in the proof is illustrated by Figure 7.6.

Let \(p = p' + \binom{n-1}{k-2}\). Then \(p' > 0\) as \(p > \binom{n-1}{k-2}\) and \(p' = p - \binom{n-1}{k-2} \leq \binom{n-2}{k}\) as \(p \leq \binom{n-1}{k-2} + \binom{n-2}{k}\). Further, \(\binom{n-2}{k-1} \leq \binom{n-1}{k-2}\) by assumption and \(\binom{n-1}{k-2} \leq \binom{n-1}{k}\) for \(k \leq n/2\) so that \(\binom{n-2}{k-1} \leq \binom{n-1}{k-2} \leq \binom{n-1}{k}\).
For the new-shadow of $L_{n, k+1}(p)$ we have

$$|\Delta_N L_{n, k+1} (p)| = |\Delta_N L_{n, k+1} \left( \binom{n}{k+1} + p' \right)|$$

$$\geq |\Delta_N L_{n-1, k} \left( \binom{n-1}{k} \right)| + |\Delta_N L_{n-2, k} (p')|$$

(by Lemma 7.8 as $\binom{n-2}{k-1} \leq \binom{n-1}{k} \leq \binom{n}{k}$)

and $p' \leq \binom{n-2}{k}$

$$= |\Delta_N L_{n-1, k} \left( \binom{n-1}{k} \right)| + |\Delta_N L_{n-1, k+1} (p')|.$$  \hspace{1cm} (7.12)

(by Lemma 2.31)

For the new-shade of $L_{n, k-1}(p)$ we write

$$|\nabla_N L_{n, k-1} (p)| = |\nabla_N L_{n, k-1} \left( \binom{n}{k-2} + p' \right)|$$

$$= |\nabla_N L_{n-1, k-2} \left( \binom{n-1}{k-2} \right)| + |\nabla_N L_{n-1, k-1} (p')|.$$  \hspace{1cm} (7.13)

(by Lemma 7.4)
Combining (7.12) and (7.13) we conclude that

\[
|\Delta NL_{n,k+1}(p)| + |\nabla NL_{n,k-1}(p)| \\
\geq |\Delta NL_{n-1,k}\left(\binom{n-1}{k-2}\right)| + |\Delta NL_{n-1,k+1}(p')| \\
+ |\nabla NL_{n-1,k-2}\left(\binom{n-1}{k-2}\right)| + |\nabla NL_{n-1,k-1}(p')| \\
> 2\left(\binom{n-1}{k-2}\right) + 2p' \\
\] (by Induction Hypothesis 7.15 as \(\binom{n-1}{k-2}, p' > 0\))

= 2p.

□

Case 3.2.2.2) below is the last case to consider when \(\frac{n+1}{3} < k \leq \frac{n}{2}\). It is subdivided into two subcases depending on whether \(\frac{k-1}{n-k} + \frac{n-k-1}{k} \geq 2\) or \(\frac{k-1}{n-k} + \frac{n-k-1}{k} < 2\). The proof of the second case is long and is given in Section 7.5.

3.2.2.2) \(\binom{n-1}{k-2} + \binom{n-2}{k} < p \leq \binom{n}{k-1}\)

3.2.2.1) \(\frac{k-1}{n-k} + \frac{n-k-1}{k} \geq 2\)

**Proposition 7.21.** Let \(n > 32\) and \(\frac{n+1}{3} < k \leq \frac{n}{2}\) be such that \(\binom{n-1}{k-2} \geq \binom{n-2}{k-1}\) and \(\frac{k-1}{n-k} + \frac{n-k-1}{k} \geq 2\). Then Theorem 7.1. (i) holds for \(\binom{n-1}{k-2} + \binom{n-2}{k} \leq p \leq \binom{n}{k-1}\).

**Proof.** The partitioning of \(L_{n,k+1}(p)\) and \(L_{n,k-1}(p)\) used in the proof is illustrated by Figure 7.7.

The proof assumes that Induction Hypothesis 7.15 holds and that Propositions 7.16 and 7.20 hold for \(n\). Note that Lemma 7.12 applies for the assumed values of \(k\) and \(n\).

By assumption \(\binom{n-1}{k-2} \geq \binom{n-2}{k-1}\) and \(p > \binom{n-1}{k-2} + \binom{n-2}{k}\) so that \(p > \binom{n-2}{k-1} + \binom{n-2}{k}\). Thus if we let \(p = p' + \binom{n-2}{k-1}\) we see that \(p' > 0\).
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\[ \binom{n}{k+1} \quad \binom{n-1}{k+1} \quad \binom{n-2}{k} \quad \binom{n-2}{k-1} \]

\[ \binom{n}{k+1} \quad \binom{n-1}{k+1} \quad \binom{n-2}{k} \quad \binom{n-2}{k-1} \]

\[ k + 1 \]

\[ p' \]

\[ p \]

\[ k - 1 \]

\[ p' \]

\[ \binom{n-2}{k-1} \]

\[ \binom{n-1}{k-2} \]

\[ \binom{n}{k-1} \]

Figure 7.7: Partitioning of \( L_{n,k+1}(p) \) and \( L_{n,k-1}(p) \) in the proof of Proposition 7.21

Also,

\[ p' = p - \binom{n-2}{k-1} \]

\[ \leq \binom{n}{k-1} - \binom{n-2}{k-1} \quad \text{(as } p \leq \binom{n}{k-1} \text{ by assumption)} \]

\[ = \binom{n-2}{k-2} + \binom{n-1}{k-2} \]

\[ \leq \binom{n-2}{k} + \binom{n-1}{k-2} \quad \text{(as } \binom{n-2}{k} \leq \binom{n-2}{k-1} \text{ for } k \leq \frac{n}{2} ) \]

Hence Propositions 7.16 and 7.20 apply to \( p' \) and the values of \( k \) considered in this proof, namely \( \frac{n+1}{3} < k \leq \frac{n}{2} \). Since \( p' > 0 \), this enables us to write that

\[ |\Delta_N L_{n,k+1}(p')| + |\nabla_N L_{n,k-1}(p')| > 2p'. \quad (7.14) \]
Further,

\[
|\Delta_N L_{n,k+1} (p)| = |\Delta_N L_{n,k+1} \left( \binom{n-2}{k-1} + p' \right)|
\]
\[
= |\Delta_N L_{n,k+1} \left( \binom{n-2}{k-1} \right)| + |\Delta_N P_{n,k+1}^{(2)} (p')| \tag{by Observation 2.11}
\]
\[
\geq |\Delta_N L_{n,k+1} \left( \binom{n-2}{k-1} \right)| + |\Delta_N L_{n,k+1} (p')|; \tag{7.15}
\]

(by Note 2.52 and Theorem 2.47)

For the new-shade of \( L_{n,k-1}(p) \) we write

\[
|\nabla_N L_{n,k-1} (p)| = |\nabla_N L_{n,k-1} \left( p' + \binom{n-2}{k-1} \right)|
\]
\[
= |\nabla_N L_{n,k-1} (p')| + |\nabla_N F_{n,k-1}^{p'} \left( \binom{n-2}{k-1} \right)| \tag{by Observations 2.9 and 2.11}
\]
\[
\geq |\nabla_N L_{n,k-1} (p')| + |\nabla_N F_{n,k-1} \left( \binom{n-2}{k-1} \right)|. \tag{7.16}
\]

(by Note 2.52 and Corollary 2.49)

(7.15) and (7.16) combined give

\[
|\Delta_N L_{n,k+1} (p)| + |\nabla_N L_{n,k-1} (p)| \]
\[
\geq |\Delta_N L_{n,k+1} \left( \binom{n-2}{k-1} \right)| + |\Delta_N L_{n,k+1} (p')| \]
\[
+ |\nabla_N L_{n,k-1} (p')| + |\nabla_N F_{n,k-1} \left( \binom{n-2}{k-1} \right)| \]
\[
> 2p' + 2 \binom{n-2}{k-1} \tag{by Lemma 7.12 and (7.14)}
\]
\[
= 2p.
\]
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7.5 The Proof of Theorem 7.1.(i): Part B

Section 7.5 deals with the case labelled 3.2.2.2.2) in the outline of the proof for Theorem 7.1.(i). This is the case when $p$ is in the range $\binom{n-1}{k-2} + \binom{n-2}{k} < p \leq \binom{n}{k-1}$ for $n > 32$, $\frac{n+1}{3} < k \leq \frac{n}{2}$, $\binom{n-1}{n-2} \geq \binom{n-2}{n-1}$, and $\frac{k+1}{n-k} + \frac{n-k+1}{k} < 2$. That Theorem 7.1.(i) holds in this case is shown by Proposition 7.22 which completes the proof of Theorem 7.1.(i) for $k \leq \frac{n}{2}$.

**Proposition 7.22.** Let $n > 32$ and $\frac{n+1}{3} < k \leq \frac{n}{2}$ with $\binom{n-1}{n-2} \geq \binom{n-2}{n-1}$ and $\frac{k+1}{n-k} + \frac{n-k+1}{k} < 2$. Then Theorem 7.1.(i) holds for $(\binom{n-1}{k-2} + \binom{n-2}{k}) < p \leq \binom{n}{k-1}$.

The proof of Proposition 7.22 requires Lemmas 7.30 and 7.31. The proof of Lemma 7.31 relies on Lemma 7.32 whose proof is long and complex: It requires Lemmas 7.36 to 7.56. The proof of these lemmas form the bulk of this section.

Subsection 7.5.1 states definitions and results which are needed throughout the section. Subsection 7.5.2 proves Lemma 7.30 and then Proposition 7.22 assuming that Lemma 7.31 holds. In Subsection 7.5.3 the proof of Lemma 7.31 is given, it assumes that Lemma 7.32 holds. The remainder of the section is devoted to proving Lemma 7.32 by induction. The base case is proved in Subsection 7.5.4, this involves proving Lemmas 7.36 to 7.39. The inductive step is proved in Subsection 7.5.5 and involves proving Lemmas 7.40 to 7.56. Recall that Induction Hypothesis 7.15 is assumed to hold.

7.5.1 Preliminaries

**Terminology.** All collections are assumed to be collections of sets in squashed order. A collection of consecutive $k$-sets is meant to be a collection of consecutive $k$-sets in **squashed order**. Whenever we say that a collection $\mathcal{D}$ of $q$ $k$-sets **comes before** (after) a collection $\mathcal{C}$ of $k$-sets, we mean that $\mathcal{D}$ consists of $q$ consecutive $k$-sets in squashed order that come immediately before (after) the first (last) set in $\mathcal{C}$ in squashed order.
The definitions given here are only relevant in the context of the present section. Recall that throughout this section, \((\frac{n-2}{k}) + (\frac{n-2}{k-1}) < p \leq (\frac{n}{k})\) with \(n > 32\), \(\frac{k+1}{3} < k \leq \frac{n}{3}\), \((\frac{n-2}{k-2})\) \(\geq (\frac{n}{k-1})\), and \(\frac{k+1}{n-k} + \frac{n-k-1}{k} < 2\).

**Definition 7.23.**

Let \(A = I_{n,k+1}(p)\) and \(B = I_{n,k-1}(p)\).

**Definition 7.24.**

For \(q \in \mathbb{Z}\), \(0 \leq q \leq \min\{\left(\frac{n}{k+1}\right), \left(\frac{n}{k-1}\right)\}\), \(1 \leq k < n\), let \(U\) be a collection of \(q\) \((k+1)\)-subsets of \([n]\) and \(D\) a collection of \(q\) \((k-1)\)-subsets of \([n]\). Then the pair \((U, D)\) has property \(P\) if \(|\triangle_N U| + |\triangledown_N D| \geq 2q\).

The following observations and lemma are needed in the remainder of the section. As \((\frac{n-1}{k-2}) \geq (\frac{n-2}{k-1})\) and \(p > (\frac{n-1}{k-2}) + (\frac{n-2}{k})\) by assumption, \(|A| = p > (\frac{n-1}{k})\). It follows that

**Observation 7.25.** \(I_{n,k+1}\left(\left(\frac{n-1}{k}\right)\right) \subseteq A\).

**Observation 7.26.** If \((U, D)\) has property \(P\), then \(|U| = |D|\).

The next two observations are consequences of Observation 2.10.

**Observation 7.27.** Assume that \(U_1\) and \(U_2\) are collections of \((k+1)\)-sets and that \(D_1\) and \(D_2\) are collections of \((k-1)\)-sets such that \(U_1 \cap U_2 = D_1 \cap D_2 = \emptyset\) and \((U_1, D_1)\) and \((U_2, D_2)\) each has property \(P\). Then \((U_1 \cup U_2, D_1 \cup D_2)\) has property \(P\).

**Observation 7.28.** Let \(\{Q_1, Q_2\}\) be a partition of a collection \(Q\) such that \(|\triangledown_N Q_1| \geq \frac{n-k}{k}|Q_1|\) and \(|\triangledown_N Q_2| \geq \frac{n-k}{k}|Q_2|\). Then \(|\triangledown_N Q| \geq \frac{n-k}{k}|Q|\).

**Lemma 7.29.** Let \(q\) be given with \(0 \leq q \leq \left(\frac{n-1}{k}\right)\). Let \(U = F\left(q, I_{n,k+1}\left(\left(\frac{n-1}{k}\right)\right)\right)\) and let \(D\) be a collection of \(q\) \((k-1)\)-sets such that \(|\triangledown_N D| \geq \frac{n-k}{k}|D|\). Then \((U, D)\) has property \(P\).

**Proof.** Note that for \(x \in \mathbb{R}^+\), the function

\[
f(x) = x + \frac{1}{x}\]

is minimum at \(f(1) = 2\). (7.17)
Let $\mathcal{U}$ and $\mathcal{D}$ be as defined in the statement of the lemma. Note that $\mathcal{U} = F_{n-1,k}(q) \cup \{n\}$. Then $\mathcal{U}$ corresponds to $F_{n-1,k}(q)$ by Lemma 2.30. It follows that

$$|\Delta_N \mathcal{U}| = |\Delta_N F_{n-1,k}(q)|$$

$$= |\Delta F_{n-1,k}(q)|$$

(by Observation 2.9)

$$\geq \frac{k}{n-k} q.$$  \hspace{1cm} (7.18)

(by Sperner’s lemma)

Then

$$|\Delta_N \mathcal{U}| + |\nabla_N \mathcal{D}| \geq \frac{k}{n-k} q + \frac{n-k}{k} q$$

(by (7.18))

$$\geq 2q.$$  \hspace{1cm} (by (7.17))

This proves the lemma. \hfill \Box

### 7.5.2 The Proof of Proposition 7.22

The Proof of Proposition 7.22 requires Lemmas 7.30 and 7.31. We state and prove Lemma 7.30 first, after which we state Lemma 7.31. Assuming that Lemma 7.31 holds, we then prove Proposition 7.22. The proof of Lemma 7.31 is deferred until Subsection 7.5.3. Recall that $\mathcal{A} = L_{n,k+1}(p)$ and $\mathcal{B} = L_{n,k-1}(p)$.

#### Lemma 7.30

Let $n > 32$, $k \leq \frac{n}{4}$, $\binom{n-1}{k-2} + \binom{n-2}{k} < p \leq \binom{n}{k-1}$, and let $\mathcal{D} = L(\binom{n-k+2}{1}, \mathcal{B})$. Then $|\nabla_N \mathcal{D}| > 2|\mathcal{D}|$.

**Proof of Lemma 7.30.** Let $\mathcal{D} = L(\binom{n-k+2}{1}, \mathcal{B})$ be as in the statement of the lemma. Note that $|\mathcal{B}| = p > \binom{n-k+2}{1}$. Note also that $\mathcal{D}$ corresponds to the collection
\[ L_{n+k+2,1} \left( \binom{n-k+2}{1} \right) \text{ by Lemma 2.31. Then} \]
\[
\left| \nabla_N D \right| = \left| \nabla_N L_{n-k+2,1} \left( \binom{n-k+2}{1} \right) \right| 
\]
\[
= \frac{n-k+1}{2} \binom{n-k+2}{1} 
\]  
(by Lemma 2.31)

\[
> 2 \binom{n-k+2}{1} 
\]  
(for \( n > 6 \) since \( k \leq \frac{n}{2} \))

as required. \( \square \)

**Lemma 7.31.** Let \( n > 32 \) and \( \frac{1}{3} \leq k \leq \frac{n}{2} \) with \( \binom{n-1}{k-2} \geq \binom{n-2}{k-1} \) and \( \frac{k+1}{n-k} + \frac{n-k-1}{k} < 2 \). Let \( p \) be such that \( \binom{n-1}{k-2} + \binom{n-2}{k} < p \leq \binom{n-1}{k-1} \). Let \( \mathcal{P}_2 = F(p - \binom{n-k+2}{1}, \mathcal{B}) \). Then there exists a collection \( \mathcal{P}_1 \subseteq \mathcal{A} \) such that \((\mathcal{P}_1, \mathcal{P}_2)\) has property \( P \).

The proof of Lemma 7.31 appears in Subsection 7.5.3. Assuming that Lemma 7.31 holds, we now prove Proposition 7.22.

**Proof of Proposition 7.22.** Assume that Lemma 7.31 holds and recall that \( |\mathcal{A}| = |\mathcal{B}| = p \). Note that \( p - \binom{n-k+2}{1} > 0 \) and let \( D = L(\binom{n-k+2}{1}, \mathcal{B}) \) and \( \mathcal{P}_2 = F(p - \binom{n-k+2}{1}, \mathcal{B}) \). Then \( \{\mathcal{P}_2, D\} \) is a partition of \( \mathcal{B} \). By Lemma 7.31 there exists a collection \( \mathcal{P}_1 \subseteq \mathcal{A} \) such that \((\mathcal{P}_1, \mathcal{P}_2)\) has property \( P \). Thus \( |\triangle N \mathcal{P}_1| + |\nabla_N \mathcal{P}_2| \geq 2|\mathcal{P}_1| = 2|\mathcal{P}_2| \). Note that \( |A \setminus \mathcal{P}_1| = |B \setminus \mathcal{P}_2| = |D| \). By Lemma 7.30, \( |\triangle N (A \setminus \mathcal{P}_1)| + |\nabla_N \mathcal{P}_2| > 2|D| = 2|A \setminus \mathcal{P}_1| \). Therefore \( |\triangle N L_{m,k+1}(p)| + |\nabla_N L_{m,k-1}(p)| = |\triangle N \mathcal{A}| + |\nabla_N \mathcal{B}| > 2p \) by Observation 2.10. This proves Proposition 7.22. \( \square \)

Note that the proof of Proposition 7.22 is similar to the proofs of the previous propositions. Proposition 7.22 is proved by finding an appropriate partitioning for the collections \( \mathcal{A} \) and \( \mathcal{B} \). Here \( \mathcal{A} \) and \( \mathcal{B} \) are partitioned into \( \{\mathcal{P}_1, \mathcal{A} \setminus \mathcal{P}_1\} \) and \( \{\mathcal{P}_2, D\} \) respectively, with \( |\mathcal{P}_1| = |\mathcal{P}_2| \) and \( |A \setminus \mathcal{P}_1| = |D| \). It is shown that \((\mathcal{P}_1, \mathcal{P}_2)\) and \((\mathcal{A} \setminus \mathcal{P}_1, D)\) each has property \( P \), with the strict inequality \( |\triangle N (A \setminus \mathcal{P}_1)| + |\nabla_N D| > 2|D| \).
It remains to prove that Lemma 7.31 holds. This is addressed next.

### 7.5.3 The Proof of Lemma 7.31

Lemma 7.32 is required to prove that Lemma 7.31 holds. Lemma 7.32 is stated first and is followed by two observations. Then, assuming that Lemma 7.32 holds, we prove Lemma 7.31. Lemma 7.32 will be proved in Subsections 7.5.4 and 7.5.5. Recall that $A = L_{n,k+1}(p)$ and $B = L_{n,k-1}(p)$.

**Lemma 7.32.** Let $n > 32$ and $\frac{n+1}{3} < k \leq \frac{n}{2}$ with $\binom{n-1}{k-1} \geq \left(\frac{k}{n-1}\right)$ and $\frac{n+1}{2} + \frac{n}{k-1} < 2$. Let $p$ be such that $\left(\binom{n-1}{k-1} + \binom{n-2}{k-1}\right) < p \leq \binom{n}{k-1}$. Let $l \in N$ be such that $0 \leq l \leq k - 4$, and let $q_i = p - \binom{n-2}{k-3} + \sum_{j=1}^{l} \left(\binom{n-3-j}{k-3-j-1}\right)$. Let $P_1(l) = F(q_i, B)$. Then for each $l$, there exists a collection $P_1(l)$ and partitions $\{S_1(l), T_1(l)\}$ and $\{S_2(l), T_2(l)\}$ of $P_1(l)$ and $P_2(l)$ respectively such that

(i) (a) $P_1(l)$ is a collection of $q_i$ consecutive $(k+1)$-sets,

(b) $P_1(l) \subseteq A$,

(ii) there exists $s \in N$ with

(a) $S_1(l) = F(s, L_{n,k+1}\left(\binom{n-1}{k}\right))$,

(b) $L(s, P_1(l)) = S_1(l)$ if $L_{n,k+1}\left(\binom{n-1}{k}\right) \setminus P_1(l) \neq \emptyset$,

(iii) $|\nabla N S_2(l)| \geq \frac{n-1}{k} |S_2(l)|$,

(iv) $(T_1(l), T_2(l))$ has property $P$.

**Note 7.33.**

To aid readability the notation $P_1(l)$, $S_1(l)$, $T_1(l)$, $P_2(l)$, $S_2(l)$, $T_2(l)$ in Lemma 7.32 will not be used and alternative notation will be defined as appropriate.

For the remainder of this subsection the notation $P_1$, $S_1$, $T_1$, $P_2$, $S_2$, $T_2$ is used to denote the collections $P_1(l)$, $S_1(l)$, $T_1(l)$, $P_2(l)$, $S_2(l)$, $T_2(l)$ in Lemma 7.32.

**Observation 7.34.** In Lemma 7.32 (ii)(a), saying that $S_1 = F(s, L_{n,k+1}\left(\binom{n-1}{k}\right))$ is equivalent to saying that $S_1 = F_{n-1,k}(s) \cup \{n\}$. 
Note that $L_{n,k+1}\left(\binom{n-1}{k}\right) \subset \mathcal{A}$ by Observation 7.25 so that $(L_{n,k+1}\left(\binom{n-1}{k}\right) \setminus \mathcal{P}_1) \subseteq (\mathcal{A} \setminus \mathcal{P}_1)$ and $|L_{n,k+1}\left(\binom{n-1}{k}\right) \setminus \mathcal{P}_1| \leq |\mathcal{A} \setminus \mathcal{P}_1| = |\mathcal{B} \setminus \mathcal{P}_2| = p - |\mathcal{P}_2|$. It follows that

**Observation 7.35.** In Lemma 7.32 (i), saying that $\mathcal{P}_1 \subseteq \mathcal{A}$ is equivalent to saying that $|L_{n,k+1}\left(\binom{n-1}{k}\right) \setminus \mathcal{P}_1| \leq p - |\mathcal{P}_2|$.

Figure 7.8 pictures the collections $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{S}_1$ satisfying the conditions of Lemma 7.32. The collections $\mathcal{A}$ and $\mathcal{B}$ are indicated by a hatched line. Note that it is assumed that $L_{n,k+1}\left(\binom{n-1}{k}\right) \setminus \mathcal{P}_1 \neq \emptyset$ when drawing Figure 7.8.

Assuming that Lemma 7.32 holds, we now prove Lemma 7.31.

**Proof of Lemma 7.31.** Assume that Lemma 7.32 holds for each $l$, $0 \leq l \leq k - 4$, and let $l = k - 4$. Then $q_l = q_{k-4} = p - \binom{n-2}{k-3} + \sum_{j=1}^{k-4} \binom{n-3-j}{k-3-j} = p - \binom{n-k+2}{k-1}$ and $\mathcal{P}_2 = F(p - \binom{n-k+1}{k-1}, \mathcal{B})$.

Note that $|\mathcal{T}_1| = |\mathcal{T}_2|$ by Lemma 7.32.(iv) and Observation 7.26. Thus $|\mathcal{S}_1| = |\mathcal{S}_2|$ and $(\mathcal{S}_1, \mathcal{S}_2)$ has property P by Lemma 7.32.(ii)(a) and (iii), and Lemma 7.29.

Hence $(\mathcal{S}_1 \cup \mathcal{T}_1, \mathcal{S}_2 \cup \mathcal{T}_2)$, that is $(\mathcal{P}_1, \mathcal{P}_2)$, has property P by Lemma 7.32.(iv) and Observation 7.27. Lemma 7.31 then follows from Lemma 7.32.(i).
As can be seen in the proof of Lemma 7.31, showing that \((P_1, P_2)\) has property P is achieved by using the fact that the two pairs \((S_1, S_2)\) and \((T_1, T_2)\) each have property P, where \(\{S_1, T_1\}\) and \(\{S_2, T_2\}\) are partitions of \(P_1\) and \(P_2\) respectively. Note that the fact that \((P_1, P_2)\) has property P is independent of the value of \(l\).

It remains to prove Lemma 7.32. This involves finding, for each \(l, 0 \leq l \leq k - 4\), a suitable collection \(P_1\) of \(q_l\) consecutive \((k + 1)\)-sets such that \(P_1\) and \(P_2\) satisfy Lemma 7.32. Note that the collections \(P_1\) and \(P_2\) will be different for each value of \(l\), so that their subsequent partitioning into \(\{S_1, T_1\}\) and \(\{S_2, T_2\}\) will also differ for different values of \(l\). The proof of Lemma 7.32 uses induction on \(l\). In Subsection 7.5.4 Lemma 7.36 shows that Lemma 7.32 holds for \(l = 0\). In Subsection 7.5.5 Lemma 7.40 proves the inductive step for Lemma 7.32. Thus, if Lemmas 7.36 and 7.40 hold, then Lemma 7.32 holds.

### 7.5.4 The Proof of Lemma 7.32: Base Case

In this subsection the notation \(P_1, S_1, \ldots, T_2\) is used to denote the collections \(P_1(0), S_1(0), \ldots, T_2(0)\) in Lemma 7.32.

**Lemma 7.36.** *Lemma 7.32 holds for \(l = 0\).*

The proof of Lemma 7.36 is split into three cases which are dealt with in Lemmas 7.37, 7.38, and 7.39 respectively. Lemma 7.37 considers values of \(p\) such that \((\binom{n-1}{k-2}) + (\binom{n-2}{k}) < p \leq (\binom{n-1}{k}) + (\binom{n-2}{k-3})\). The remaining values of \(p\), \((\binom{n-1}{k}) + (\binom{n-2}{k-3}) < p \leq \binom{n}{k-1}\), are considered in Lemmas 7.38 and 7.39 depending on whether \((\binom{n-2}{k}) - (\binom{n-2}{k-2}) \leq \binom{n}{k-3}\) or \((\binom{n-2}{k}) - (\binom{n-2}{k-2}) > \binom{n}{k-3}\). If Lemmas 7.37, 7.38 and 7.39 hold, then it follows that Lemma 7.36 holds.

**Lemma 7.37.** *Assume that \((\binom{n-1}{k-2}) + (\binom{n-2}{k}) < p \leq (\binom{n-1}{k}) + (\binom{n-2}{k-3})\). Then Lemma 7.32 holds for \(l = 0\).*

*Proof.* The collections \(P_1\) and \(P_2\) and their partitioning are illustrated by Figure 7.9.
When \( l = 0 \), \( q_0 = p - \binom{n-2}{k-3} \). By assumption \( p > \binom{n-1}{k-1} + \binom{n-2}{k-3} \) so \( q_0 = p - \binom{n-2}{k-3} > \binom{n-2}{k-3} + \binom{n-2}{k-3} \). Also, \( p \leq \binom{n-1}{k} + \binom{n-2}{k-3} \), so \( q_0 \leq \binom{n-1}{k} \). Let \( P_2 = F(q_0, B) \) and let \( P_1 \) denote the collection of the first \( q_0 \) \((k + 1)\)-sets which contain \( n \). That is, 
\[
P_1 = F(q_0, L_{n,k+1}(\binom{n-1}{k})).
\]

By Observation 7.25, \( L_{n,k+1}(\binom{n-1}{k}) \subseteq \mathcal{A} \), so that \( P_1 \subseteq \mathcal{A} \) and Lemma 7.32.(i) holds.

Let \( S_1 = P_1 \) and \( S_2 = P_2 \) so that \( T_1 = T_2 = \emptyset \). Then (ii) and (iv) of Lemma 7.32 hold. It is easy to see that
\[
\left\{ L_{n-1,k-1} \left( q_0 - \binom{n-2}{k-2} \right), L_{n-2,k-2} \left( \binom{n-2}{k-2} \right) \right\} \cup \{n\} \text{ is a partition of } S_2.
\]

(7.19)

It follows that
\[
|\nabla_N S_2| = |\nabla_N L_{n-1,k-1} \left( q_0 - \binom{n-2}{k-2} \right)| + |\nabla_N L_{n-2,k-2} \left( \binom{n-2}{k-2} \right)|.
\]

(by (7.19)), Observation 2.10

and Lemma 2.30)
Thus

\[
\left| \nabla \mathcal{S}_2 \right| = \left| \nabla L_{n-1,k-1} \left( q_0 - \binom{n-2}{k-2} \right) \right| + \left| \nabla L_{n-2,k-2} \left( \binom{n-2}{k-2} \right) \right| \\
\geq \frac{n-k}{k} \left( q_0 - \binom{n-2}{k-2} \right) + \frac{n-k}{k-1} \binom{n-2}{k-2} \\
= \frac{n-k}{k} |\mathcal{S}_2|
\]

(by Observation 2.9) 

\[
\geq \frac{n-k}{k} q_0
\]

(by Sperner’s lemma) 

Therefore Lemma 7.32.(iii) holds. This completes the proof. \[\Box\]

**Lemma 7.38.** Assume that \( \binom{n}{k} + \binom{n-2}{k-3} < p \leq \binom{n}{k-1} \) and \( \binom{n-2}{k-3} - \binom{n-2}{k-2} \leq \binom{n-2}{k-3} \). Then Lemma 7.32 holds for \( l = 0 \).

**Proof.** The collections \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) and their partitioning are illustrated by Figure 7.10.

When \( l = 0 \), \( q_0 = p - \binom{n-2}{k-3} \). By assumption, \( p > \binom{n-1}{k} + \binom{n-2}{k-3} \) so \( q_0 = p - \binom{n-2}{k-3} \).

![Figure 7.10: The collections \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) in the proof of Lemma 7.38](image-url)
\[
\binom{n-1}{k} \geq \binom{n-1}{k-1} \leq \binom{n-1}{k} \quad \text{for } k \leq \frac{n}{2}.
\]

Let \( \mathcal{P}_2 = F(g_0, B) \) and let \( \{ \mathcal{S}_2, \mathcal{T}_2 \} \) be a partition of \( \mathcal{P}_2 \) such that \( \mathcal{S}_2 = F\left(\binom{n-1}{k-1}, \mathcal{P}_2\right) \).

The aim is to find a collection \( \mathcal{P}_1 \subseteq \mathcal{A} \) such that Lemma 7.32 holds for \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \).

Let \( \mathcal{P}_1 = \mathcal{S}_1 \cup \mathcal{T}_1 \) where \( \{ \mathcal{S}_1, \mathcal{T}_1 \} \) is a partition of \( \mathcal{P}_1 \), and \( \mathcal{S}_1 = F\left(\binom{n-1}{k}, L_{n,k+1}\left(\binom{n-1}{k}\right)\right) \) and \( \mathcal{T}_1 = L(g_0 - \binom{n-1}{k-1}, F_{n,k+1}\left(\binom{n-1}{k+1}\right)) \). Then Lemma 7.32.(ii) holds. Also,

\[ T_1 \text{ is the collection } L_{n-1,k+1}\left(q_0 - \frac{n-1}{k-1}\right). \quad (7.20) \]

by Observation 2.6 applied to \( F_{n,k+1}\left(\binom{n-1}{k+1}\right) \).

Note that \( \mathcal{P}_1 \) is a collection of \( g_0 \) consecutive \((k+1)\)-sets. We show that \( \mathcal{P}_1 \subseteq \mathcal{A} \). By Observation 7.35 it is enough to show that \( |L_{n,k+1}\left(\binom{n-1}{k}\right) \setminus \mathcal{P}_1| \leq p - |\mathcal{P}_2| \). We have

\[
\begin{align*}
|L_{n,k+1}\left(\binom{n-1}{k}\right) \setminus \mathcal{P}_1| &= |L_{n,k+1}\left(\binom{n-1}{k}\right) \setminus (\mathcal{S}_1 \cup \mathcal{T}_1)| \\
&= |L_{n,k+1}\left(\binom{n-1}{k}\right) \setminus \mathcal{S}_1| \\
&= \binom{n-1}{k} - \binom{n-1}{k-1} \\
&= \binom{n-2}{k} - \binom{n-2}{k-2} \\
&\leq \binom{n-2}{k-3} \quad \text{(as } \binom{n-2}{k} - \binom{n-2}{k-2} \leq \binom{n-2}{k-3} \text{ by assumption)} \\
&= p - |\mathcal{P}_2|. \\
&= p - (\binom{n-2}{k-3}) \quad \text{(as } |\mathcal{P}_2| = p - (\binom{n-2}{k-3}) \text{)}
\end{align*}
\]

It follows that Lemma 7.32.(i) holds. We now consider the new-shade of \( \mathcal{S}_2 \). Recall
Recall that 

\[ T_n \] 

is by Lemma 7.20 and it follows that 

\[ \text{Lemma 7.20} \] 

(iii) holds. It remains to show that \( (T_1, T_2) \) has property P. By assumption \( p \leq \binom{n}{k-1} \), so that 

\[ p - \binom{n-2}{k-3} - \binom{n-1}{k-1} \leq \binom{n}{k-1} - \binom{n-2}{k-3} - \binom{n-1}{k-1} = \binom{n-1}{k-2} - \binom{n-2}{k-3} = \binom{n-2}{k-2}. \] 

As \( \binom{n-2}{k-2} \leq \binom{n-2}{k} \) for \( k \leq \frac{n}{2} \), it follows that \( q_0 - \binom{n-1}{k-1} \leq \binom{n-1}{k-2} \) and that 

\[ q_0 - \binom{n-1}{k-1} \leq \binom{n-2}{k}. \] 

Recall that \( T_2 = L(q_0 - \binom{n-1}{k-1}, \mathcal{P}_2) \). Since \( q_0 - \binom{n-1}{k-1} \leq \binom{n-2}{k-2} \), it is not difficult to see that \( T_2 \) is the collection \( L_{n-2,k-2}(q_0 - \binom{n-1}{k-1}) \oplus \{ n \} \). Thus 

\[ T_2 \text{ corresponds to } L_{n-2,k-2} \left( q_0 - \binom{n-1}{k-1} \right) \] 

(7.21)

by Lemma 2.30 and it follows that

\[ |\triangle_N T_1| + |\nabla N T_2| \]

\[ = |\nabla N L_{n-1,k+1} \left( q_0 - \binom{n-1}{k-1} \right) | + |\nabla N L_{n-2,k-2} \left( q_0 - \binom{n-1}{k-1} \right) |. \]

(by 7.20 and 7.21)
Thus

\[
|\triangle_N T_1| + |\nabla_N T_2| = |\triangle_N L_{n-2,k} (q_0 - \binom{n-1}{k-1})| + |\nabla_N L_{n-2,\kappa} - \binom{n-1}{k-1})|
\]

(by Lemma 2.31 as \(q_0 - \binom{n-1}{k-1}) \leq \binom{n-2}{k-2}\))

\[> 2 \left( q_0 - \binom{n-1}{k-1}) \right).
\]

(by Induction Hypothesis 7.15)

as \(q_0 - \binom{n-1}{k-1}) > 0\)

Therefore \((T_1, T_2)\) has property P and Lemma 7.32(iv) holds. This concludes the proof. \(\square\)

**Lemma 7.39.** Assume that \(\binom{n-1}{k} + \binom{n-2}{k-3} < p \leq \binom{n}{k-1}\) and \(\binom{n-2}{k-2} > \binom{n-2}{k-3}\).

Then Lemma 7.32 holds for \(l = 0\).

**Proof.** The collections \(P_1\) and \(P_2\) and their partitioning are illustrated by Figure 7.11. As \(\frac{k-1}{n-k} + \frac{n-k-1}{k} < 2\) and \(\binom{n-2}{k-2} > \binom{n-2}{k-3}\), Lemmas E.4 and E.5 apply.

When \(l = 0\), \(q_0 = p - \binom{n-2}{k-3}\). By assumption, \(p \geq \binom{n-1}{k-1} + \binom{n-2}{k-3}\) so \(q_0 = p - \binom{n-2}{k-3} > \binom{n-1}{k-1}\). Let \(m = \binom{n-1}{k-1} + \binom{n-3}{k-2}\) and note that \(m \leq \binom{n-1}{k-1}\) by Lemma E.4. This implies that \(q_0 - m > 0\).

Let \(P_2 = F(q_0, B)\) and let \(\{S_2, T_2\}\) be a partition of \(P_2\) such that \(S_2 = F(m, P_2)\). The aim is to find a collection \(P_1 \subseteq A\) such that Lemma 7.32 holds for \(P_1\) and \(P_2\).

Let \(P_1 = L_{n-1, k} \cup \{S_1, T_1\}\) where \(\{S_1, T_1\}\) is a partition of \(P_1\), and \(S_1 = F(m, L_{n, k+1} \left(\binom{n-1}{k}\right))\) and \(T_1 = L(q_0 - m, F_{n,k+1} \left(\binom{n-1}{k+1}\right))\). Then Lemma 7.32(ii) holds. Also,

\[
T_1 \text{ is the collection } L_{n-1,k+1} (q_0 - m)
\]

(7.22)

by Observation 2.6 applied to \(F_{n,k+1} \left(\binom{n-1}{k+1}\right)\).

Note that \(P_1\) is a collection of \(q_0\) consecutive \((k+1)\)-sets. We show that \(P_1 \subseteq A\). By
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Figure 7.11: The collections $\mathcal{P}_1$ and $\mathcal{P}_2$ in the proof of Lemma 7.39

Observation 7.35 it is enough to show that $|L_{n,k+1}(\binom{n-1}{k}) \setminus \mathcal{P}_1| \leq p - |\mathcal{P}_2|$. 

$$
\begin{align*}
|L_{n,k+1}(\binom{n-1}{k}) \setminus \mathcal{P}_1| &= |L_{n,k+1}(\binom{n-1}{k}) \setminus (S_1 \cup T_1)| \\
&= |L_{n,k+1}(\binom{n-1}{k}) \setminus S_1| \\
&= \binom{n-1}{k} - \binom{n-1}{k-1} - \binom{n-3}{k-2} \\
&= \binom{n-2}{k} - \binom{n-2}{k-2} - \binom{n-3}{k-2} \\
&\leq \binom{n-2}{k-3} \\
&\quad \text{(by Lemma E.5)} \\
&\quad \text{(as $T_1 \cap L_{n,k+1}(\binom{n-1}{k}) = \emptyset$)} \\
&= p - |\mathcal{P}_2|. \\
&\quad \text{(as $|\mathcal{P}_2| = p - \binom{n-2}{k-3}$)}
\end{align*}
$$

It follows that Lemma 7.32.(i) holds. We now consider the new-shade of $S_2$. Recall
that \( S_2 \) is a collection \( C_{n,k-1}((\binom{n-1}{k-1}) + (\binom{n-3}{k-2})) \). Then,

\[
\nabla_N S_2 = \left| \nabla_N C_{n,k-1} \left( \binom{n-1}{k-1} + \binom{n-3}{k-2} \right) \right| \\
\geq \left| \nabla_N F_{n,k-1} \left( \binom{n-1}{k-1} + \binom{n-3}{k-2} \right) \right|
\]

(by Corollary 2.49)

\[
\nabla_N S_2 = \left| \nabla_N F_{n-1,k-1} \left( \binom{n-1}{k-1} \right) \right| + \left| \nabla_N F_{n-3,k-2} \left( \binom{n-3}{k-2} \right) \right|
\]

(by Observation 2.10

and Lemmas 2.28 and 2.30)

\[
= \left| \nabla_N L_{n-1,k-1} \left( \binom{n-1}{k-1} \right) \right| + \left| \nabla_N L_{n-3,k-2} \left( \binom{n-3}{k-2} \right) \right|
\]

(by Observation 2.7)

\[
\geq \frac{n-k}{k} \left( \binom{n-1}{k-1} + \binom{n-3}{k-2} \right)
\]

(by Observation 2.9 and Lemma 2.35)

\[
\geq \frac{n-k}{k} |S_2|
\]

(1) holds. It remains to show that \((T_1, T_2)\) has property P. By assumption, \( p \leq \frac{n}{k-1} \) so that \( p - \binom{n-2}{k-3} - m \leq \frac{n}{k-1} - \binom{n-2}{k-3} - \binom{n-1}{k-2} - \binom{n-3}{k-2} = \binom{n-3}{k-2} \).

As \( \binom{n-2}{k-3} \leq \binom{n-3}{k-2} \) for \( k \leq \frac{n}{k-2} \), it follows that \( q_0 - m \leq \binom{n-3}{k-2} \) and that \( q_0 - m \leq \binom{n-2}{k-3} \).

Recall that \( T_2 = L(q_0 - m, T_2) \). Since \( q_0 - m \leq \binom{n-3}{k-2} < \binom{n-2}{k-2} \) it is not difficult to see that \( T_2 \) is the collection \( L_{n-2,k-2}(q_0 - m) \oplus \{n\} \). Thus

\[
T_2 \text{ corresponds to } L_{n-2,k-2}(q_0 - m)
\]
Thus

\[ |\triangle N T_1| + |\triangledown N T_2| \]
\[ = |\triangle N L_{n-3,k-1} (q_0 - m)| + |\triangledown N L_{n-3,k-3} (q_0 - m)| \]
\[ \quad \text{(by Lemma 2.31 as } q_0 - m \leq \binom{n-3}{k-1} \text{)} \]
\[ \quad \text{and } q_0 - m \leq \binom{n-3}{k-1} \]
\[ > 2 (q_0 - m) . \]
\[ \quad \text{(by Induction Hypothesis 7.15} \]
\[ \quad \text{as } q_0 - m > 0) \]

Therefore \((T_1, T_2)\) has property P and Lemma 7.32(iv) holds. This concludes the proof.

\[ \square \]

### 7.5.5 The Proof of Lemma 7.32: Inductive Step

For the remainder of the proof of Lemma 7.32 the collections \(P_1(l), S_1(l), \ldots, T_2(l)\) in Lemma 7.32 are denoted by \(P'_1, S'_1, \ldots, T'_2\) respectively when \(l = i\) and by \(P_1, S_1, \ldots, T_2\) respectively when \(l = i + 1\). Note that here the dash does not carry any intrinsic meaning and is used for notational convenience only.

**Lemma 7.40.** If Lemma 7.32 holds for \(l = i\) then Lemma 7.32 holds for \(l = i + 1\) for all \(0 \leq i < k - 4\).

The proof of Lemma 7.40 requires the consideration of six cases. The proof is presented in three parts. Part A summarises what is known if Lemma 7.32 holds for \(l = i\) and states what is required to prove that Lemma 7.32 holds for \(l = i + 1\). Part B is a collection of four lemmas which are needed in Part C where the six cases are outlined and where it is shown that Lemma 7.32 holds for \(l = i + 1\) in each of the six cases.

Assume that Lemma 7.32 holds for \(l = i\). For \(l = i + 1\), \(P_2 = F(q_{i+1}, B)\). To prove that Lemma 7.32 holds for \(l = i + 1\) one must find a collection \(P_1\) of \(q_{i+1}\) consecutive \((k + 1)\)-sets so that (i), (ii), (iii), and (iv) of Lemma 7.32 hold for \(P_1\) and \(P_2\). This
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involves finding appropriate partitions \( \{S_1, T_1\} \) and \( \{S_2, T_2\} \) of \( P_1 \) and \( P_2 \) respectively. It will be seen in the proof of Lemma 7.40 that the partitions of \( P_1 \) and \( P_2 \) (when \( l = i + 1 \)) are built using the partitions of \( P'_1 \) and \( P'_2 \) (when \( l = i \)).

Proof of Lemma 7.40: Part A

The induction hypothesis in the proof is

**Induction Hypothesis 7.41.** Assume that Lemma 7.32 holds for \( l = i \).

We show that Lemma 7.32 holds for \( l = i + 1 \), that is, when \( P_2 = F(q_{i+1}, B) \) where \( q_{i+1} = p - \binom{n-2}{k-3} + \sum_{j=1}^{i+1} \binom{n-3 -(j-1)}{k_3-(j-1)} \). The following definitions, Definitions 7.42 to 7.44 are used in this proof only. They define collections of sets which are used throughout the proof.

Definition 7.42 defines collections of sets for \( l = i \). If Induction Hypothesis 7.41 holds then the collections \( P'_1, S'_1, T'_1, P'_2, S'_2, \) and \( T'_2 \) in Definition 7.42 exist. For convenience, Definition 7.42 also includes a repetition of Definition 7.23 and of the hypotheses in Proposition 7.22 and Lemmas 7.32 and 7.40.

**Definition 7.42.**

Let \( n, k \) be such that \( n > 32, \frac{n-1}{3} < k \leq \frac{n}{2}, \binom{n-1}{k-2} \geq \binom{n-2}{k-1}, \) and \( \frac{n-1}{n-k} + \frac{n-k-1}{k} < 2 \).

Let \( p \) be such that \( \binom{n-1}{k-2} + \binom{n-2}{k} < p \leq \binom{n}{k-1} \).

Let \( A = L_{n,k+1}(p) \) and \( B = L_{n,k-1}(p) \).

Let \( i \in \mathbb{N} \) be such that \( 0 \leq i < k - 4 \).

Let \( q_i = p - \binom{n-2}{k-3} + \sum_{j=1}^{i} \binom{n-3 -(j-1)}{k_3-(j-1)} \).

Let \( q_{i+1} = p - \binom{n-2}{k-3} + \sum_{j=1}^{i+1} \binom{n-3 -(j-1)}{k_3-(j-1)} \).

Let \( P'_2 = F(q_i, B) \).
Definition 7.42 (continued).

Let $\mathcal{P}_1'$ be a collection of sets with $\{S'_1, T'_1\}$ and $\{S'_2, T'_2\}$ partitions of $\mathcal{P}_1'$ and $\mathcal{P}_2'$ respectively such that

(i) $\mathcal{P}_1'$ is a collection of $q_i$ consecutive $(k+1)$-sets and $\mathcal{P}_1' \subseteq \mathcal{A},$

(ii) there exists $s' \in \mathcal{N}$ such that

(a) $S'_1 = F(s', L_{n,k+1}(\binom{n-1}{k})),$

(b) $L(s', \mathcal{P}_1') = S'_2$ if $L_{n,k+1}(\binom{n-1}{k}) \setminus \mathcal{P}_1' \neq \emptyset,$

(iii) $|\nabla_N S'_2| \geq \frac{n-k}{k} |S'_2|,$

(iv) $(T'_1, T'_2)$ has property P.

Let $m_i = |L_{n,k+1}(\binom{n-1}{k}) \setminus \mathcal{P}_1'|.$

Definition 7.43 defines collections of $(k-1)$-sets for $l = i+1.$ In particular, it defines a collection of $q_{i+1}$ $(k-1)$-sets.

**Definition 7.43.**

Let $\mathcal{X}$ and $\mathcal{P}_2$ be collections of $(k-1)$-sets such that

(i) (a) $\mathcal{X}$ comes after $\mathcal{P}_2',$

(b) $|\mathcal{X}| = \binom{n-3-i}{k-3-i},$

(ii) $\mathcal{P}_2 = \mathcal{P}_2' \cup \mathcal{X}.$

Definition 7.44 concerns collections of $(k+1)$-sets for $l = i+1$ for which some properties are stated. These collections of $(k+1)$-sets will be fully defined later in Lemmas 7.51 to 7.56.

**Definition 7.44.**

Let $\mathcal{L}, \mathcal{R},$ and $\mathcal{P}_1$ be collections of $(k+1)$-sets such that

(i) $\mathcal{L}$ comes before $\mathcal{P}_1',$

(ii) $\mathcal{R}$ comes after $\mathcal{P}_1',$

(iii) $|\mathcal{L}| + |\mathcal{R}| = \binom{n-3-i}{k-3-i},$

(iv) $\mathcal{P}_1 = \mathcal{L} \cup \mathcal{P}_1' \cup \mathcal{R}.$

Figure 7.5.5 illustrates the collections $\mathcal{P}_1', \mathcal{P}_2', \mathcal{L}, \mathcal{R}, \mathcal{X}, \mathcal{P}_1,$ and $\mathcal{P}_2$ as they are given.
in Definitions 7.42 to 7.44. Note that it is assumed that $m_i > 0$ when drawing Figure 7.5.5.

Figure 7.12: The collections $\mathcal{P}', \mathcal{P}'', \mathcal{L}, \mathcal{R}, \mathcal{X}, \mathcal{P}_1,$ and $\mathcal{P}_2$ in Definitions 7.42, 7.43, and 7.44

Assume that Induction Hypothesis 7.41 holds. It is easy to see that $\mathcal{P}_2 = F(q_{i+1}, B)$ and that $|\mathcal{P}_1| = q_{i+1}$. Thus, to prove that Lemma 7.32 holds for $l = i + 1$, it is sufficient to prove that Lemma 7.32 holds for $\mathcal{P}_1$ and $\mathcal{P}_2$ as given in Definitions 7.43 and 7.44. To do so it is necessary to find appropriate collections $\mathcal{L}$ and $\mathcal{R}$ having the properties outlined in Definition 7.44. Once $\mathcal{L}$ and $\mathcal{R}$ are known, $\mathcal{P}_1$ is then fully defined. It remains to prove that $\mathcal{P}_1 \subseteq \mathcal{A}$ and that partitions $\{S_1, T_1\}$ and $\{S_2, T_2\}$ of
$P_1$ and $P_2$ respectively can be found in a way that ensures that Lemma 7.32 holds. Six cases will be considered, each of them discussing a different value for $m_i$ (see Lemmas 7.51 to 7.56). These cases are outlined and proved in Part C of the proof (see Figure 7.13).

For convenience while writing the proof of Lemma 7.40 we state the following observations which follow from Definitions 7.42 to 7.44. All subsequent proofs will then refer to these observations. Observation 7.45 summarises what is known about $P_1'$, $P_2'$, $S'_1$, $S'_2$, $T'_1$, and $T'_2$, while Observation 7.46 summarises what is known about $P_1$, $P_2$, $L$, $R$, and $X$. Justification for these observations is given after Observation 7.46.

**Observation 7.45.**

(i) $p - |P'_2| = \binom{n-2-i}{k-3-i}$,

(ii) $m_i = |L_{n,k+1}\left(\binom{n-1}{k}\right) \setminus P'_1|$ and $m_i \leq p - |P'_2|$,

(iii) $\{S'_1, T'_1\}$ and $\{S'_2, T'_2\}$ are partitions of $P'_1$ and $P'_2$ respectively,

(iv) $S'_1 = F(s', L_{n,k+1}\left(\binom{n-1}{k}\right))$,

(v) $|\nabla N S'_2| \geq \frac{n-k}{k} |S'_2|$,

(vi) $(T'_1, T'_2)$ has property P.

**Observation 7.46.**

(i) $L$ comes before $P'_1$, $R$ comes after $P'_1$, $X$ comes after $P'_2$, and $|R| + |L| = |X|$,

(ii) $L$ and $R$ correspond to some collection $C_{n-1,k+1}(|L|)$ and $C_{n-1,k}(|R|)$ respectively, and $X$ corresponds to $L_{n-2-i,k-3-i}\left(\binom{n-1-i}{k-3-i}\right)$,

(iii) $\{L, P'_1, R\}$ and $\{P'_2, X\}$ are partitions of $P_1$ and $P_2$ respectively,

(iv) $P_1$ is a collection of $q_{i+1}$ consecutive $(k+1)$-sets,

(v) $P_2 = F(q_{i+1}, B)$,

(vi) $|R| \leq m_i$.

Note that $p - |P'_2| = p - q_i = \binom{n-2-i}{k-3-i}$. Thus Observation 7.45.(i) holds. Observation 7.45.(ii) follows from Induction Hypothesis 7.41 and Observation 7.35. The remaining points of Observation 7.45 follow from Induction Hypothesis 7.41.

Observations 7.46 (i) and (iii) repeat Definitions 7.43 and 7.44. Since $q_{i+1} = q_i +
(\binom{n-3-i}{k-3-i}) (iv) and (v) of Observation 7.46 hold. Observe that \(\mathcal{L}\) corresponds to some collection \(C_{n-1,k+1}(|\mathcal{L}|)\), and that \(\mathcal{R}\) corresponds to some collection \(C_{n-1,k}(|\mathcal{R}|)\) by Lemma 2.30 as all sets in \(\mathcal{R}\) contain the element \(n\). In addition, \(\mathcal{X}\) is the collection \(F_{n-3-i,k-3-i}(\binom{n-3-i}{k-3-i})\{n-i-1, n-i, \ldots, n\}\). Then \(\mathcal{X} = L_{n-3-i,k-3-i}(\binom{n-3-i}{k-3-i})\{n-i-1, n-i, \ldots, n\}\) by Observation 2.7. Hence \(\mathcal{X}\) corresponds to \(L_{n-3-i,k-3-i}(\binom{n-3-i}{k-3-i})\) by Lemma 2.30. It follows that Observation 7.46.(ii) holds.

Finally, Observation 7.46.(vi) follows from the fact that \(m_i\) denotes the maximum number of \((k+1)\)-sets which come after \(\mathcal{P}_1\). This is the case since \(m_i = |L_{n,k+1}(\binom{n-1}{k})|\) with \(S'_1 \subseteq \mathcal{P}_1'\) and \(S'_1 = F(s', L_{n,k+1}(\binom{n-1}{k}))\) by Observations 7.45.(iii) and (iv).

(End of Part A of the proof of Lemma 7.40)

**Proof of Lemma 7.40: Part B**

We state and prove the four lemmas, Lemmas 7.47 to 7.50, which are required to show that Lemma 7.40 holds in the six cases outlined and proved in Part C of the proof. Lemma 7.47 outlines sufficient conditions under which Lemma 7.32.(i) holds for \(\mathcal{P}_1\), while Lemma 7.48 outlines sufficient conditions under which (ii), (iii), and (iv) of Lemma 7.32 hold for \(\mathcal{P}_1\) and \(\mathcal{P}_2\). The conditions in both lemmas are expressed in terms of the collections of sets defined in Definitions 7.42 to 7.44.

**Lemma 7.47.** Assume that one of the following conditions holds:

(a) \(|\mathcal{R}| = \binom{n-3-i}{k-3-i}\), or

(b) \(m_i \leq \binom{n-3-i}{k-4-i}\) and \(|\mathcal{R}| = 0\), or

(c) \(m_i < \binom{n-3-i}{k-3-i}\) and \(|\mathcal{R}| = \binom{n-4-i}{k-5-i}\), or

(d) \(m_i < \binom{n-4-i}{k-4-i}\) and \(|\mathcal{R}| = \binom{n-5-i}{k-6-i}\), or

(e) \(m_i < \binom{n-5-i}{k-3-i}\) and \(|\mathcal{R}| = \binom{n-6-i}{k-6-i}\).

Then \(\mathcal{P}_1 \subseteq \mathcal{A}\).

That is, Lemma 7.32.(i) holds for \(\mathcal{P}_1\).
Proof. We show that in each case, $|L_{n,k+1} \left( \binom{n-1}{k} \right) \setminus P_1| \leq p - |P_2|$. By Observation 7.35 this is equivalent to showing that $P_1 \subseteq A$. As

$$|L_{n,k+1} \left( \binom{n-1}{k} \right) \setminus P_1| = |L_{n,k+1} \left( \binom{n-1}{k} \right) \setminus P_1'| - |R|$$

(as, by Observations 7.45.(iii), (iv) and 7.46.(i)),

$$L \cap L_{n,k+1} \left( \binom{n-1}{k} \right) = \emptyset,$$

$$R \subseteq L_{n,k+1} \left( \binom{n-1}{k} \right), \text{ and } P_1' \cap R = \emptyset$$

$$= m_i - |R|,$$

(by Observation 7.45.(ii))

the proof is completed by showing that $m_i - |R| \leq p - |P_2|$.

For Case (a) $m_i - |R| \leq p - |P_2'| - |R|$ by Observation 7.45.(ii). Note that $|R| = |X|$ by Observation 7.46.(ii). Thus $m_i - |R| \leq p - |P_2'| - |R| = p - |P_2'| - |X| = p - |P_2|$ by Observation 7.46.(iii).

For Case (b) $m_i - |R| \leq \binom{n-3-i-j}{k-3-i-j}$. For Cases (c), (d), and (e), $m_i - |R| < \binom{n-3-i-j}{k-3-i-j} - \binom{n-4-i-j}{k-4-i-j} < \binom{n-3-i-j}{k-4-i-j}$ with $j = 0, 1, 2$. Note that $p - |P_2| = p - |P_2'| - |X| = \binom{n-2-i-j}{k-3-i-j} - \binom{n-3-i-j}{k-3-i-j}$ by Observations 7.46.(iii) and (ii) and 7.45.(i). Therefore $m_i - |R| \leq p - |P_2|$ in Cases (b), (c), (d), and (e).

It follows that in each case, $P_1 \subseteq A$. Note that $P_1$ is a collection of $q_{i+1}$ consecutive $(k+1)$-sets by Observation 7.46.(iv). This shows that Lemma 7.32.(i) holds for $P_1$. \qed

In Lemma 7.48 below it is shown how a partitioning of $P_1$ and $P_2$ into $\{S_1, T_1\}$ and $\{S_2, T_2\}$ respectively can be made from a partitioning of $P_1'$ and $P_2'$ into $\{S_1', T_1'\}$ and $\{S_2', T_2'\}$ respectively such that (ii), (iii), and (iv) of Lemma 7.32 hold.

**Lemma 7.48.** Assume that $(x_R, x_C)$ is a partition of $X$ with $x_R$ coming before $x_C$. Assume that $\{S_1, T_1\}$ and $\{S_2, T_2\}$ are partitions of $P_1$ and $P_2$ respectively and assume that one of the following conditions holds:
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(a) $S_1 = S'_1 \cup R$,
$T_1 = T'_1 \cup L$,
$S_2 = S'_2 \cup X_R$,
$T_2 = T'_2 \cup X_L$,

with $|\nabla_N X_R| \geq \frac{n-k}{k} |X_R|$ and $(L, X_L)$ having property $P$,
or
(b) $S_1 = S'_1$,
$T_1 = T'_1 \cup L \cup R$,
$S_2 = S'_2$,
$T_2 = T'_2 \cup X$,

with $S'_1 \cup R = L_{n,k+1} \left( \binom{n-1}{k} \right)$ and $(L \cup R, X)$ having property $P$.

Then the collections $P_1$, $S_1$, $T_1$, $S_2$, and $T_2$ have the following properties:

(i) there exists $s \in N$ with

(a) $S_1 = F(s, L_{n,k+1} \left( \binom{n-1}{k} \right))$,

(b) $L(s, P_1) = S_1$ if $L_{n,k+1} \left( \binom{n-1}{k} \right) \setminus P_1 \neq \emptyset$,

(ii) $|\nabla_N S_2| \geq \frac{n-k}{k} |S_2|$,

(iii) $(T_1, T_2)$ has property $P$.

That is, (ii), (iii), and (iv) of Lemma 7.32 hold for $P_1$, $S_1$, $T_1$ and $P_2$, $S_2$, $T_2$.

Proof. We prove each of (i), (ii), and (iii) in turn.

(i) We consider two cases.

(1) $m_i = 0$. Then $R = \emptyset$ by Observation 7.46(vi). It follows that $S_1 = S'_1 = F(s', L_{n,k+1} \left( \binom{n-1}{k} \right))$ by Observation 7.45(iv), and that $L_{n,k+1} \left( \binom{n-1}{k} \right) \setminus P_1 = L_{n,k+1} \left( \binom{n-1}{k} \right) \setminus P'_1 = \emptyset$ by Observations 7.46(iii) and (i) and Observation 7.45(iv).

(2) $m_i > 0$. 

For Case (a), \( S_1 = S'_1 \cup R \), and by Observation 7.45.(v), \( L(s', P'_1) = S'_1 \), so that \( R \) comes after \( S'_1 \) by Observation 7.46.(1). Hence, by Observation 7.45.(iv), the collection \( S_1 = S'_1 \cup R \) is the collection \( F(s' + |R|, L_{n,k+1}\binom{n-1}{k}) \). That is, \( S_1 = F(s, L_{n,k+1}\binom{n-1}{k}) \) where \( s = s' + |R| \). Note that

\[
L_{n,k+1}\binom{n-1}{k} \cap P_1 \text{ are the last } L_{n,k+1}\binom{n-1}{k} \cap P_1 \text{ sets of } P_1, \tag{7.24}
\]

as \( P_1 \) consists of consecutive \((k+1)\)-sets by Observation 7.46.(iv), that

\[
L_{n,k+1}\binom{n-1}{k} \cap P'_1 = S'_1 \tag{7.25}
\]

by Observations 7.45.(iv) and (v), and that \( \{S'_1, R\} \) is a partition of \( S_1 \).

Then \( L_{n,k+1}\binom{n-1}{k} \cap P_1 = (L_{n,k+1}\binom{n-1}{k} \cap P'_1) \cup (L_{n,k+1}\binom{n-1}{k} \cap R) = S'_1 \cup R = S_1 \) by (7.25) and by the fact that \( R \subseteq L_{n,k+1}\binom{n-1}{k} \) by Observation 7.46.(i). By (7.24) this is equivalent to say that \( L(s, P_1) = S_1 \). Therefore Lemma 7.32.(iii)(b) holds for \( P_1 \) and \( S_1 \).

For Case (b), \( S_1 = S'_1 = F(s', L_{n,k+1}\binom{n-1}{k}) \) by Observation 7.45.(iv). By assumption, \( S'_1 \cup R = L_{n,k+1}\binom{n-1}{k} \). This implies that \( L_{n,k+1}\binom{n-1}{k} \setminus P_1 = \emptyset \) since \((S_1 \cup R) \subseteq P_1 \) by Observation 7.46.(iii).

(ii) For Case (a), note that \( |\nabla_N X_R| \geq \frac{\alpha k}{2}|X_R| \). \( \{S'_2, X_R\} \) is a partition of \( S_2 \) by Observation 7.46.(i). Then, by Observation 7.28 together with Observation 7.45.(vi), we conclude that \( |\nabla_N S_2| \geq \frac{\alpha k}{2}|S_2| \).

In Case (b) the result follows from Observation 7.45.(vi).

(ii) For Case (a), note that \( (L, X_L) \) has property P and that \( \{T'_1, L\} \) and \( \{T'_2, X_L\} \) are partitions of \( T_1 \) and \( T_2 \) respectively by Observation 7.46.(i). For Case (b), note that \( (L \cup R, X) \) has property P and that \( \{T'_1, L \cup R\} \) and \( \{T'_2, X\} \) are partitions of \( T_1 \) and \( T_2 \) respectively by Observation 7.46.(i). In both cases we conclude that \( (T_1, T_2) \) has property P by Observations 7.45.(vii) and 7.27.

This concludes the proof of Lemma 7.48.

The next two lemmas are the last results in Part B of the proof of Lemma 7.40.
They are required in the proofs of the lemmas in Part C to enable the application of Lemma 7.48. The collection \( \mathcal{X} \) in Lemma 7.49 is as in Definition 7.43.

**Lemma 7.49.** Assume that \((\mathcal{X}_R, \mathcal{X}_L)\) is a partition of \( \mathcal{X} \) with \( \mathcal{X}_R \) coming before \( \mathcal{X}_L \). Assume that one of the following conditions holds:

(a) \( \mathcal{X}_R \) corresponds to the collection \( L_{n-3-i,k-3-i}(|\mathcal{X}_R|) \), or

(b) \( \mathcal{X}_R \) corresponds to the collection \( L_{n-4-i,k-3-i}(|\mathcal{X}_R|) \), or

(c) \( \mathcal{X}_R \) corresponds to the collection \( L_{n-5-i,k-3-i}(|\mathcal{X}_R|) \), or

(d) \( \mathcal{X}_R \) corresponds to the collection \( L_{n-6-i,k-3-i}(|\mathcal{X}_R|) \) and \((\binom{n-3-i}{k-3-i}) < (\binom{n-5-i}{k-3-i})\).

Then \( |\nabla_N \mathcal{X}_R| \geq \frac{n-k}{k} |\mathcal{X}_R| \).

**Proof.** In each case, \( \mathcal{X}_R \) corresponds to the collection \( L_{n-3-i,j,k-3-i}(|\mathcal{X}_R|) \) for \( j = 0, 1, 2, 3 \). Then

\[
|\nabla_N \mathcal{X}_R| = |\nabla_N L_{n-3-i,j,k-3-i}(|\mathcal{X}_R|)| = \frac{n-k-j}{k-2-i} |\mathcal{X}_R|.
\]

(by Observation 2.9 and Lemma 2.35)

We show that \( \frac{n-k-j}{k-2-i} \geq \frac{n-k}{k} \) for \( j = 0, \ldots, 3 \).

For \( j = 0, 1, 2 \), it is not difficult to see that \( \frac{n-k-j}{k-2-i} \geq \frac{n-k}{k} \) when \( k \leq \frac{n}{2} \). When \( j = 3 \), the additional condition \((\binom{n-3-i}{k-3-i}) < (\binom{n-5-i}{k-3-i})\) applies. Thus Lemma E.7 applies. From this we conclude that \( \frac{n-k-j}{k-2-i} \geq \frac{n-k}{k} \). This proves the lemma.

\( \square \)

Lemma 7.50 refers to general collections of sets and not just the particular collections in Definitions 7.42 to 7.44.

**Lemma 7.50.** Let \( \mathcal{U} \) and \( \mathcal{D} \) be collections of consecutive \((k+1)\)-sets and \((k-1)\)-sets respectively. Assume that one of the following conditions holds:

(a) \( \mathcal{U} \) corresponds to a collection \( C_{n-1,k+1}(\binom{n-3-i}{k-3-i}) \) and \( \mathcal{D} \) corresponds to the collection \( L_{n-3-i,k-3-i}(\binom{n-3-i}{k-3-i}) \), or
(b) \( U \) corresponds to a collection \( C_{n-1,k+1}((n-k-4-i)) \) and \( D \) corresponds to the collection \( L_{n-4-i,k-4-i}((n-k-4-i)) \), or

(c) \( U \) corresponds to a collection \( C_{n-1,k+1}((n-k-5-i)) \) and \( D \) corresponds to the collection \( L_{n-5-i,k-4-i}((n-k-5-i)) \), and \( (n-k-3-i) < (n-k-5-i) \), or

(d) \( U \) corresponds to a collection \( C_{n-1,k+1}((n-k-6-i)) \) and \( D \) corresponds to the collection \( L_{n-6-i,k-4-i}((n-k-6-i)) \), and \( (n-k-3-i) < (n-k-6-i) \).

Then \((U, D)\) has property \( P \).

**Proof.** In each case we have

\[
\left| \nabla N D \right| = \left| \nabla N L_{n-3-i,j,k-3-i,l} \left( \binom{n-3-i-j}{k-3-i-l} \right) \right| \tag{7.26}
\]

where either \( j = 0 \) and \( l = 0 \), or \( j, l = 1 \) and \( l = 1 \).

For Cases (a) and (b) we have

\[
\left| \Delta N U \right| = \left| \Delta N C_{n-1,k+1} \left( \binom{n-3-i-j}{k-3-i-l} \right) \right| \geq \left| \Delta N L_{n-1,k+1} \left( \binom{n-3-i-j}{k-3-i-l} \right) \right| \tag{7.27}
\]

(by Theorem 2.47)

\[
= \left| \Delta N L_{n-3-i-j,k-1-i,l} \left( \binom{n-3-i-j}{k-3-i-l} \right) \right| \tag{7.27}
\]

(by Lemma 2.31)

where either (1) \( j = l = 0 \), or (2) \( j = l = 1 \). Recall that \( k \leq \frac{n}{2} \). We verify that Lemma 2.31 can be applied in (7.27):

(1) For \( j = l = 0 \), \( (n-k-3-i) \leq (n-k-1-i) \).

(2) For \( j = l = 1 \), \( (n-k-4-i) \leq (n-k-2-i) \).

Now, \( |\Delta N U| + |\nabla N D| \geq 2|D| = 2|U| \) by (7.26), (7.27), and Induction Hypothesis 7.15.

For Case (c) note that as \( U \) corresponds to \( C_{n-1,k+1}((n-k-4-i)) \) then, by Lemma 2.30, \( U \)
corresponds to $C_{n-1,k+1}\left(\binom{n-5-i}{k-4-i}\right) \uplus \{n\}$ which is a collection $C_{n,k+1}\left(\binom{n-5-i}{k-4-i}\right)$. Then,

$$|\Delta N U| = \left| \Delta_N C_{n,k+1}\left(\binom{n-5-i}{k-4-i}\right) \right|$$

$$\geq \left| \Delta_N L_{n,k+1}\left(\binom{n-5-i}{k-4-i}\right) \right|$$

(by Theorem 2.47)

$$\geq \left| \Delta_N L_{n-5-i,k-4-i}\left(\binom{n-5-i}{k-4-i}\right) \right|$$

(by Lemma 2.31)

$$= \left(\frac{n-5-i}{k-5-i}\right).$$

(7.28)

By (7.26), and Observations 2.9, 2.7 and 2.8,

$$|\nabla_N D| = \left| \nabla_N L_{n-5-i,k-4-i}\left(\binom{n-5-i}{k-4-i}\right) \right| = \left(\frac{n-5-i}{k-3-i}\right).$$

(7.29)

In Case (c), the condition $\binom{n-5-i}{k-4-i} < \binom{n-4-i}{k-3-i}$ holds. Thus Lemma E.6 applies and $\binom{n-5-i}{k-5-i} + \binom{n-5-i}{k-3-i} \geq 2\binom{n-5-i}{k-4-i}$. It follows that, by (7.28), (7.29), and Lemma E.6,

$$|\nabla_N D| + |\Delta_N U| \geq 2|D| = 2|U|.$$

The proof for Case (d) is very similar to that of Case (c). For Case (d) we have

$$|\Delta_N U| = \left| \Delta_N C_{n-1,k+1}\left(\binom{n-6-i}{k-4-i}\right) \right|$$

$$\geq \left| \Delta_N L_{n-1,k+1}\left(\binom{n-6-i}{k-4-i}\right) \right|$$

(by Theorem 2.47)

$$\geq \left| \Delta_N L_{n-6-i,k-4-i}\left(\binom{n-6-i}{k-4-i}\right) \right|$$

(by Lemma 2.31)

$$= \left(\frac{n-6-i}{k-5-i}\right).$$

(7.30)

(by Observations 2.9, 2.7 and 2.8)

By (7.26), and Observations 2.9, 2.7 and 2.8,

$$|\nabla_N D| = \left| \nabla_N L_{n-6-i,k-4-i}\left(\binom{n-6-i}{k-4-i}\right) \right| = \left(\frac{n-6-i}{k-3-i}\right).$$

(7.31)
In Case (d), the condition $\binom{n-3-i}{k-3-i} < \binom{n-5-i}{k-3-i}$ holds. Thus Lemma E.8 applies and $\binom{n-3-i}{k-3-i} + \binom{n-5-i}{k-3-i} \geq 2\binom{n-4-i}{k-3-i}$. It follows that, by (7.30), (7.31), and Lemma E.8, $|\nabla_N P| + |\nabla_N U| \geq 2|P| = 2|U|$. This concludes the proof. 

(End of Part B of the proof of Lemma 7.40)

Proof of Lemma 7.40: Part C

In the remainder of the proof of Lemma 7.40 the collections of sets referred to are as in Definitions 7.42 to 7.44. Six cases need to be considered in order to prove that Lemma 7.40 holds (see Figure 7.13). Each case considers a different value of $m_i$. Recall that $m_i = |L_{n,k+1}\left(\binom{n-1}{k}\right)\setminus P'_1|$. By Observation 7.46.(vi) the size of $R$ is dependent on $m_i$. Therefore, for each value of $m_i$, a different collection $R$ has to be defined and thus a different way of forming $P_1$ must be found. Further, in each case an appropriate partitioning of $P_1$ and $P_2$ into $\{S_1, T_1\}$ and $\{S_2, T_2\}$ respectively must be found so that Lemma 7.32 is satisfied. The results in Part B are used for each proof in this part. The six cases are outlined in Figure 7.13 and for each case the lemma that pertains to it is indicated. Recall also that $0 \leq i < k - 4$.

1) $m_i \geq \binom{n-3-i}{k-3-i}$ \hspace{1cm} (Lemma 7.51)

2) $m_i < \binom{n-3-i}{k-3-i}$

2.1) $m_i \leq (\binom{n-3-i}{k-3-i})$ \hspace{1cm} (Lemma 7.52)

2.2) $m_i > (\binom{n-3-i}{k-3-i})$

2.2.1) $m_i \geq (\binom{n-4-i}{k-3-i})$ \hspace{1cm} (Lemma 7.53)

2.2.2) $m_i < (\binom{n-4-i}{k-3-i})$

2.2.2.1) $m_i \geq (\binom{n-5-i}{k-3-i})$ \hspace{1cm} (Lemma 7.54)

2.2.2.2) $m_i < (\binom{n-5-i}{k-3-i})$

2.2.2.2.1) $m_i \geq (\binom{n-6-i}{k-3-i})$ \hspace{1cm} (Lemma 7.55)

2.2.2.2.2) $m_i < (\binom{n-6-i}{k-3-i})$ \hspace{1cm} (Lemma 7.56)

Figure 7.13: Outline of the cases considered in the proof of Lemma 7.40
When proving the six lemmas the numbering of each case under consideration will follow the numbering in Figure 7.13. Lemmas 7.51 to 7.56 complete the proof of Lemma 7.40. That is, Lemmas 7.51 to 7.56 prove that if Lemma 7.32 holds for \( l = i \), then Lemma 7.32 holds for \( l = i + 1, 0 \leq i < k - 4 \), since this is the assertion in Lemma 7.40. Therefore, if Lemmas 7.51 to 7.56 hold then Lemma 7.32 holds, as Lemma 7.36 shows that Lemma 7.32 holds for \( l = 0 \). The truth of Lemma 7.32 implies that Lemma 7.31 holds, and consequently that Proposition 7.22 holds.

For ease of reading, each of Lemmas 7.51 to 7.56 will refer to Lemma 7.32 instead of Lemma 7.40. Note that if Lemma 7.32 is said to hold for \( P_1 \) and \( P_2 \), then Lemma 7.32 holds for \( l = i + 1 \) by Observations 7.46.(iv) and (v). Recall that Induction Hypothesis 7.41 is assumed to hold. That is, it is assumed that Lemma 7.32 holds for \( l = i \).

In each of Lemmas 7.51 to 7.56 the proof is accompanied by a supporting figure visualising how \( P_1 \) and \( P_2 \) are formed. For illustrative purposes the value of \( i \) chosen in these figures is \( i = 0 \) and the situation pictured for \( i = 0 \) is that of Lemma 7.37.

Recall that Lemma 7.37 considers the situation when \( P_1' = S_1' \) and \( P_2' = S_2' \) so that \( T_1' = T_2' = \emptyset \). In this case, \( P_2' = F(p - \binom{n-3}{k-3}, B) \), and \( \mathcal{X} \) is the collection of \( \binom{n-3}{k-3} \) sets that comes after \( P_2' \). The collections \( \mathcal{A} \) and \( \mathcal{B} \) are indicated by hatched lines, \( P_1' \) and \( P_2' \) are represented by bold lines. The other collections shown are \( \mathcal{X}, \mathcal{R} \) and \( \mathcal{L} \). None of \( S_1, S_2, T_1 \) and \( T_2 \) are shown, so as not to clutter the figures.

1) \( m_i \geq \binom{n-3-i}{k-3-i} \)

**Lemma 7.51.** Let \( m_i \geq \binom{n-3-i}{k-3-i} \). Then Lemma 7.32 holds for \( P_1 \) and \( P_2 \). That is, if Lemma 7.32 holds for \( l = i \) then Lemma 7.32 holds for \( l = i + 1 \).

**Proof.** The collections \( P_1 \) and \( P_2 \) and their partitioning are illustrated by Figure 7.14.

Let \( |\mathcal{R}| = \binom{n-3-i}{k-3-i} \), \( |\mathcal{L}| = 0 \). As \( |\mathcal{R}| = \binom{n-3-i}{k-3-i} \), Lemma 7.32.(i) holds by Lemma 7.47.

Let

\[
S_1 = S_1' \cup \mathcal{R}, \quad S_2 = S_2' \cup \mathcal{X}, \quad T_1 = T_1', \quad \text{and} \quad T_2 = T_2'.
\]
By Observation 7.46.(ii) \( A' \) corresponds to the collection \( L_{n-3-i,k-3-i}(|V|) \). By Lemma 7.49 it follows that

\[
|\nabla_{A'} V| \geq \frac{n-k}{k}|V|.
\]  

(7.33)

Then (7.32) and (7.33) together with Lemma 7.48 imply that (ii), (iii), and (iv) of Lemma 7.32 hold. Thus Lemma 7.32 holds for \( P_1 \) and \( P_2 \).

2) \( m_i < \binom{n-3-i}{k-3-i} \)

2.1) \( m_i \leq \binom{n-3-i}{k-4-i} \)

**Lemma 7.52.** Let \( m_i < \binom{n-3-i}{k-3-i} \) and \( m_i \leq \binom{n-3-i}{k-4-i} \). Then Lemma 7.32 holds for \( P_1 \) and \( P_2 \). That is, if Lemma 7.32 holds for \( l = i \) then Lemma 7.32 holds for \( l = i + 1 \).
Proof. The collections $\mathcal{P}_1$ and $\mathcal{P}_2$ and their partitioning are illustrated by Figure 7.15.

![Diagram](image)

Figure 7.15: The collections $\mathcal{P}_1$ and $\mathcal{P}_2$ in the proof of Lemma 7.52

Let $|R| = 0$, $|\mathcal{L}| = \binom{n-3}{k-3}$. As $m_i \leq \binom{n-3}{k-3}$ and $|R| = 0$, Lemma 7.32.(i) holds by Lemma 7.47. Let

$$S_1 = S'_1, \quad S_2 = S'_2, \quad \mathcal{T}_1 = \mathcal{T}'_1 \cup \mathcal{L}, \quad \text{and} \quad \mathcal{T}_2 = \mathcal{T}'_2 \cup \mathcal{X}. \quad (7.34)$$

By Observation 7.46.(ii), $\mathcal{L}$ and $\mathcal{X}$ correspond to a collection $C_{n-1,k+1} \left( \binom{n-3}{k-3} \right)$ and the collection $L_{n-3-i,k-3-i} \left( \binom{n-3}{k-3} \right)$ respectively. It follows that

$$\left( \mathcal{L}, \mathcal{X} \right) \text{ has property P} \quad (7.35)$$

by Lemma 7.50. Then (7.34) and (7.35) together with Lemma 7.48 imply that (ii), (iii), and (iv) of Lemma 7.32 hold. Thus Lemma 7.32 holds for $\mathcal{P}_1$ and $\mathcal{P}_2$. \qed
2.2) \( m_i > \binom{n-3-i}{k-4-i} \)

2.2.1) \( m_i \geq \binom{n-4-i}{k-3-i} \)

Lemma 7.53. Let \( \binom{n-7-i}{k-4-i} < m_i < \binom{n-3-i}{k-3-i} \) and \( m_i \geq \binom{n-4-i}{k-3-i} \). Then Lemma 7.32 holds for \( P_1 \) and \( P_2 \). That is, if Lemma 7.32 holds for \( l = i \) then Lemma 7.32 holds for \( l = i+1 \).

Proof. The collections \( P_1 \) and \( P_2 \) and their partitioning are illustrated by Figure 7.16.

<table>
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<tr>
<th>( \binom{n}{k+1} )</th>
<th>( \binom{n-1}{k+1} )</th>
<th>( \binom{n-1}{k} )</th>
</tr>
</thead>
<tbody>
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<td>( P_1' )</td>
<td>( \mathcal{R} )</td>
</tr>
<tr>
<td>( \mathcal{P}_2' )</td>
<td>( \mathcal{X}_\mathcal{R} )</td>
<td>( \mathcal{X}_\mathcal{L} )</td>
</tr>
<tr>
<td>( \binom{n-4}{k-3} )</td>
<td>( \binom{n-4}{k-4} )</td>
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<td>( \binom{n-2}{k-2} )</td>
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<td>( \binom{n-1}{k-1} )</td>
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</tr>
</tbody>
</table>

Figure 7.16: The collections \( P_1 \) and \( P_2 \) in the proof of Lemma 7.53

Let \( |\mathcal{R}| = \binom{n-4-i}{n-3-i} \), \( |\mathcal{L}| = \binom{n-4-i}{n-4-i} \). Let \( \{\mathcal{X}_\mathcal{R}, \mathcal{X}_\mathcal{L}\} \) be the partition of \( \mathcal{X} \) with \( \mathcal{X}_\mathcal{R} = F(\binom{n-4-i}{n-3-i}, \mathcal{X}) \). It follows that \( |\mathcal{X}_\mathcal{L}| = \binom{n-4-i}{n-4-i} \).
As \( m_i < \binom{n-4}{k-3-i} \) and \( |R| = \binom{n-4}{k-3-i} \), Lemma 7.32(i) holds by Lemma 7.47. Let
\[
S_1 = S_1' \cup R, \quad T_1 = T_1' \cup L, \quad S_2 = S_2' \cup X_R, \quad \text{and} \quad T_2 = T_2' \cup X_C.
\] (7.36)

By Observations 7.46(ii) and 2.7, and Lemma 2.28, \( X_R \) corresponds to the collection \( L_{n-4-i,k-3-i}([X_R]) \). By Lemma 7.49 it follows that
\[
|\nabla X_R| \geq \frac{n-k}{k} |X_R|.
\] (7.37)

By Observation 7.46(ii) and Lemmas 2.28 and 2.31, \( L \) and \( X_C \) correspond to a collection \( C_{n-1,k+1}(\binom{n-4}{k-4-i}) \) and the collection \( L_{n-4-i,k-4-i}(\binom{n-4}{k-4-i}) \) respectively. It follows that
\[
(L, X_C) \text{ has property P}
\] (7.38)

by Lemma 7.50. Then (7.36), (7.37) and (7.38) together with Lemma 7.48 imply that (ii), (iii), and (iv) of Lemma 7.32 hold. Thus Lemma 7.32 holds for \( P_1 \) and \( P_2 \).

\[\Box\]

2.2.2) \( m_i < \binom{n-4}{k-3-i} \)

2.2.2.1) \( m_i \geq \binom{n-5}{k-3-i} \)

**Lemma 7.54.** Let \( \binom{n-5}{k-3-i} < m_i < \binom{n-4}{k-3-i} \) and \( m_i \geq \binom{n-5}{k-3-i} \). Then Lemma 7.32 holds for \( P_1 \) and \( P_2 \). That is, if Lemma 7.32 holds for \( l = i \) then Lemma 7.32 holds for \( l = i + 1 \).

**Proof.** The collections \( P_1 \) and \( P_2 \) and their partitioning are illustrated by Figure 7.17.

Let \( |R| = \binom{n-5}{n-3-i} \) and \( |L| = \binom{n-5}{n-4-i} + \binom{n-4}{n-4-i} \). Let \( \{X_R, X_C\} \) be the partition of \( X \) with \( X_R = F(\binom{n-5}{n-3-i}, X) \). It follows that \( |X_C| = \binom{n-5}{n-4-i} + \binom{n-4}{n-4-i} \).

As \( m_i < \binom{n-4}{k-3-i} \) and \( |R| = \binom{n-5}{k-3-i} \), Lemma 7.32(i) holds by Lemma 7.47. Let
\[
S_1 = S_1' \cup R, \quad T_1 = T_1' \cup L, \quad S_2 = S_2' \cup X_R, \quad \text{and} \quad T_2 = T_2' \cup X_C.
\] (7.39)
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Figure 7.17: The collections $P_1$ and $P_2$ in the proof of Lemma 7.54

By Observations 7.46.(ii) and 2.7, and Lemma 2.28, $X_R$ corresponds to the collection $L_{n-5-i, k-3-i}(|X_R|)$. By Lemma 7.49 it follows that

$$\left| \nabla_N X_R \right| \geq \frac{n - k}{k} |X_R|.$$  \hspace{1cm} (7.40)

We show next that $(L, X_C)$ has property P. Let $\{L_1, L_2\}$ be a partition of $L$ such that $L_1$ and $L_2$ each consist of consecutive $(k+1)$-sets, with $|L_1| = \binom{n-5-i}{k-3}$ and $|L_2| = \binom{n-4-i}{k-4}$. Note that $L_2$ may come before $L_1$ or vice versa.

Let $\{X_1, X_2\}$ be the partition of $X_C$ with $X_1 = F(\binom{n-5-i}{k-4-i}, X_C)$. It follows that $|X_2| = \binom{n-4-i}{k-4-i}$.

By Observations 7.46.(ii) and 2.7, and Lemmas 2.28 and 2.31, $L_1$ and $X_1$ correspond to a collection $C_{n-1, k+1}(\binom{n-5-i}{k-4-i})$ and the collection $L_{n-5-i, k-4-i}(\binom{n-5-i}{k-4-i})$ respectively.
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Note that \( \binom{n-3-i}{k-3-i} < \binom{n-4-i}{k-4-i} \). It follows that

\[
(L_1, \mathcal{X}_1) \text{ has property } P
\]

by Lemma 7.50. The arguments used in the proof of Lemma 7.53, applied in the context of that proof to \( L \) and \( \mathcal{X}_L \), can be used to show that

\[
(L_2, \mathcal{X}_2) \text{ has property } P.
\]

As \( \{L_1, L_2\} \) and \( \{X_1, X_2\} \) are partitions of \( L \) and \( \mathcal{X}_L \) respectively, then, by (7.41), (7.42), and Observation 7.27, we conclude that

\[
(L, \mathcal{X}_L) \text{ has property } P.
\]

Then (7.39), (7.40) and (7.43) together with Lemma 7.48 imply that (ii), (iii), and (iv) of Lemma 7.32 hold. Thus Lemma 7.32 holds for \( P_1 \) and \( P_2 \).

\[
\text{2.2.2.2) } m_i < \binom{n-5-i}{k-3-i} \]

\[
\text{2.2.2.1) } m_i \geq \binom{n-6-i}{k-3-i}
\]

**Lemma 7.55.** Let \( \binom{n-3-i}{k-4-i} < m_i < \binom{n-4-i}{k-4-i} \) and \( m_i \geq \binom{n-5-i}{k-4-i} \). Then Lemma 7.32 holds for \( P_1 \) and \( P_2 \). That is, if Lemma 7.32 holds for \( l = i \) then Lemma 7.32 holds for \( l = i + 1 \).

**Proof.** The collections \( P_1 \) and \( P_2 \) and their partitioning are illustrated by Figure 7.18 which is very similar to Figure 7.17. Only the size of the collections \( \mathcal{X}_R, \mathcal{X}_1, \mathcal{X}_2, \mathcal{R}, \mathcal{L}_1 \) and \( \mathcal{L}_2 \) change. Let \( |\mathcal{R}| = \binom{n-6-i}{n-3-i}, |\mathcal{L}| = \binom{n-6-i}{n-4-i} + \binom{n-5-i}{n-4-i} + \binom{n-4-i}{n-3-i}. \)

Let \( \{\mathcal{X}_R, \mathcal{X}_L\} \) be the partition of \( \mathcal{X} \) with \( \mathcal{X}_R = F(\binom{n-6-i}{n-3-i}, \mathcal{X}) \). It follows that \( |\mathcal{X}_L| = \binom{n-6-i}{n-4-i} + \binom{n-5-i}{n-4-i} + \binom{n-4-i}{n-3-i}. \)

As \( m_i < \binom{n-5-i}{k-3-i} \) and \( |\mathcal{R}| = \binom{n-6-i}{k-4-i} \), Lemma 7.32-(i) holds by Lemma 7.47. Let

\[
S_1 = S_1' \cup \mathcal{R}, \quad T_1 = T_1' \cup \mathcal{L}, \quad S_2 = S_2' \cup \mathcal{X}_R, \quad \text{and} \quad T_2 = T_2' \cup \mathcal{X}_L.
\]
By Observations 7.46.(ii) and 2.7, and Lemma 2.28, $X_R$ corresponds to the collection $L_{n-6-k-3-i}(|X_R|)$. Note that $\binom{n-3-i}{k-3} < \binom{n-5-i}{k-3}$. It follows that

$$|\nabla_X X_R| \geq \frac{n-k}{k}|X_R|$$

(7.45)

by Lemma 7.49. We need to show that $(L, X_C)$ has property P. Let $\{L_1, L_2\}$ be a partition of $L$ such that $L_1$ and $L_2$ each consist of consecutive $(k+1)$-sets, with $|L_1| = \binom{n-6-i}{n-4-i}$ and $|L_2| = \binom{n-5-i}{n-4-i} + \binom{n-4-i}{n-4-i}$. Note that $L_2$ may come before $L_1$ or vice versa.

Let $\{X_1, X_2\}$ be the partition of $X_C$ with $X_1 = F(\binom{n-6-i}{n-4-i}, X_C)$. It follows that $|X_2| = \binom{n-5-i}{n-4-i} + \binom{n-4-i}{n-4-i}$.

By Observations 7.46.(ii) and 2.7, and Lemmas 2.28 and 2.31, $L_1$ and $X_1$ correspond to a collection $C_{n-1,k+1}(\binom{n-6-i}{n-4-i})$ and the collection $L_{n-6-k-4-i}(\binom{n-6-i}{n-4-i})$ respectively.
It follows that

\[(L_1, \mathcal{X}_1) \text{ has property } P\]  \hspace{1cm} (7.46)

by Lemma 7.50 since \(\binom{n-3-i}{k-3-i} < \binom{n-5-i}{k-3-i}\). The arguments used in the proof of Lemma 7.54, applied in the context of that proof to \(L\) and \(\mathcal{X}_L\), can be used to show that

\[(L_2, \mathcal{X}_2) \text{ has property } P.\]  \hspace{1cm} (7.47)

As \(\{L_1, L_2\}\) and \(\{X_1, X_2\}\) are partitions of \(L\) and \(X_L\) respectively, then, by (7.46), (7.47), and Observation 7.27, we conclude that

\[(L, \mathcal{X}_L) \text{ has property } P.\]  \hspace{1cm} (7.48)

Then (7.44), (7.45), (7.48) together with Lemma 7.48 imply that (ii), (iii), and (iv) of Lemma 7.32 hold. Thus Lemma 7.32 holds for \(P_1\) and \(P_2\).

\[\Box\]

2.2.2.2.1 \(m_i < \binom{n-6-i}{k-3-i}\)

**Lemma 7.56.** Let \(\binom{n-3-i}{k-3-i} < m_i < \binom{n-6-i}{k-3-i}\). Then Lemma 7.32 holds for \(P_1\) and \(P_2\). That is, if Lemma 7.32 holds for \(l = i\) then Lemma 7.32 holds for \(l = i + 1\).

**Proof.** The collections \(P_1\) and \(P_2\) and their partitioning are illustrated by Figure 7.19.

We are in a very similar situation to the one in Lemma 7.55. Let \(|\mathcal{R}| = m_i\), \(|L| = \binom{n-3-i}{k-3-i} - m_i\). \(X_R, X_L\) are as in the proof of Lemma 7.55. Let

\[S_1 = S_1', T_1 = T_1' \cup L \cup \mathcal{R}, S_2 = S_2', \text{ and } T_2 = T_2' \cup X_L \cup X_R.\]  \hspace{1cm} (7.49)

Note that, as \(m_i > 0\), by Observations 7.45.(iv) and (v), and 7.46.(i),

\[S_1 \cup \mathcal{R} = L_{n, k+1} \left( \binom{n-1}{k} \right).\]  \hspace{1cm} (7.50)

Observation 7.46.(iii) and (7.50) imply that \(|L_{n, k+1} \left( \binom{n-1}{k} \right) \setminus P_1| = 0 \leq p - |P_2|\). Thus \(P_1 \subseteq \mathcal{A}\) by Observation 7.35 and Lemma 7.32.(i) holds.
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Figure 7.19: The collections $\mathcal{P}_1$ and $\mathcal{P}_2$ in the proof of Lemma 7.56

We need to show that $(\mathcal{L} \cup \mathcal{R}, \mathcal{X})$ has property P. Let $\{\mathcal{L}_1, \mathcal{L}_2\}$ be a partition of $\mathcal{L}$ such that $\mathcal{L}_1$ and $\mathcal{L}_2$ each consist of consecutive $(k+1)$-sets, with $|\mathcal{L}_1| = \binom{n-6}{k-4} - m_i$ and $|\mathcal{L}_2| = \binom{n-6}{k-4} + \binom{n-5}{k-3} + \binom{n-4}{k-2}$. It follows that $|\mathcal{L}_2| = |\mathcal{X}_2|$. Note that $\mathcal{L}_2$ may come before $\mathcal{L}_1$ or vice versa.

The arguments developed in the proof of Lemma 7.56, applied in the context of that proof to $\mathcal{L}$ and $\mathcal{X}_2$, can be used to show that

$$(\mathcal{L}_2, \mathcal{X}_2)$$ has property P. \hfill (7.51)
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Note that Lemma E.9 applies since \( \binom{n-3-i}{k-4-i} < \binom{n-6-i}{k-3-i} \). Now,

\[
|\nabla_N \mathcal{X}_R| = \left| \nabla_N L_{n-6-i,k-3-i} \left( \binom{n-6-i}{k-3-i} \right) \right|
\]

(by Observations 7.46 (ii) and 2.7 and Lemma 2.28)

\[
= \binom{n-6-i}{k-2-i}
\]

(by Observations 2.9 and 2.8)

\[
\geq 2 \binom{n-6-i}{k-3-i}
\]

(by Lemma E.9)

\[
= 2|\mathcal{X}_R|.
\]

This implies that

\[
|\Delta_N L_1| + |\Delta_N R| + |\nabla_N \mathcal{X}_R| \geq 2|\mathcal{X}_R|.
\]

(7.52)

Thus \( (L_1 \cup R, \mathcal{X}_R) \) has property P.

As \( \{L_1, L_2, R\} \) and \( \{\mathcal{X}_L, \mathcal{X}_R\} \) are partitions of \( L \cup R \) and \( \mathcal{X} \) respectively, then, by (7.51), (7.52), and Observation 7.27, we conclude that

\[
(L \cup R, \mathcal{X}) \text{ has property P.}
\]

(7.53)

Then (7.49) and (7.53) together with Lemma 7.48 imply that (ii), (iii), and (iv) of Lemma 7.32 hold. Thus Lemma 7.32 holds for \( P_1 \) and \( P_2 \).

\[\square\]

(End of Part C of the proof of Lemma 7.40)

This ends the proof of Lemma 7.40. As remarked on Page 139 at the beginning of Part C of the proof of Lemma 7.40, this also ends the proof of Proposition 7.22.
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7.6 The Proof of Theorem 7.1.(i): Part C

4) $k > \frac{n}{2}$

Recall that Induction Hypothesis 7.15 is assumed to hold. For $k > \frac{n}{2}$ we begin by considering values of $p$ in the range $1 \leq p \leq \binom{n-1}{k}$. A simple induction argument will suffice in this case.

4.1 $1 \leq p \leq \binom{n-1}{k}$

Proposition 7.57. Let $n > 32$ and $k > \frac{n}{2}$. Then Theorem 7.1.(i) holds for $p \leq \binom{n-1}{k}$.

Proof. $L_{n,k+1}(p)$ and $L_{n,k-1}(p)$ in the proof are illustrated by Figure 7.20.

![Figure 7.20: The collections $L_{n,k+1}(p)$ and $L_{n,k-1}(p)$ in the proof of Proposition 7.57](image)

Note that $\binom{n-1}{k} \leq \binom{n-1}{k-2}$ for $k > \frac{n}{2}$. Thus $p \leq \binom{n-1}{k-2}$ as $p \leq \binom{n-1}{k}$ by assumption. It
follows that
\[
\left| \triangle N L_{n,k+1} (p) \right| + \left| \nabla N L_{n,k-1} (p) \right| \\
= \left| \triangle N L_{n-1,k} (p) \right| + \left| \nabla N L_{n-1,k-2} (p) \right| \\
\text{(by Lemma 2.31)} \\
\text{as } p \leq \binom{n-1}{k} \text{ and } p \leq \binom{n-1}{k-2}\]
\[> 2p.\]
\text{(by Induction Hypothesis 7.15 as } p > 0) \]

For the values of \( p > \binom{n-1}{k} \) two subcases are considered depending on whether \( \binom{n-2}{k-3} > \binom{n-1}{k} \) or \( \binom{n-2}{k-3} \leq \binom{n-1}{k} \). Note that \( \min \left\{ \binom{n}{k+1}, \frac{n}{k-1} \right\} = \frac{n}{k+1} \) for \( k > \frac{n}{2} \).

4.2. (\( \binom{n-1}{k} < p \leq \binom{n}{k+1} \))

4.2.1. (\( \binom{n-2}{k-3} > \binom{n-1}{k} \))

**Proposition 7.58.** Let \( n > 32 \) and \( k > \frac{n}{2} \) with \( \binom{n-2}{k-3} > \binom{n-1}{k} \). Then Theorem 7.1. (i) holds for \( \binom{n-1}{k} < p \leq \binom{n}{k+1} \).

**Proof.** The partitioning of \( L_{n,k+1}(p) \) and \( L_{n,k-1}(p) \) used in the proof is illustrated by Figure 7.21. Note that \( \frac{n-1}{k+1} < \frac{n-2}{k-2} \) for all values of \( k \) such that \( \frac{n}{2} < k < n \). Since \( \binom{n-1}{k} < \binom{n-2}{k-3} \) by assumption, it follows that \( \frac{n}{k+1} (\binom{n-1}{k}) < \frac{n}{k+1} (\binom{n-2}{k-3}) < \frac{n-1}{k-2} (\binom{n-2}{k-3}) \). That is, \( \binom{n}{k+1} < \binom{n-1}{k} \). Hence \( p < \binom{n-1}{k-2} \) as \( p \leq \binom{n}{k+1} \).
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Figure 7.21: Partitioning of $L_{n,k+1}(p)$ and $L_{n,k-1}(p)$ in the proof of Proposition 7.58

Let $p = p' + \binom{n-1}{k}$, so that $p' > 0$ as $p > \binom{n-1}{k}$. Then,

$$|\Delta_N L_{n,k+1}(p)| = |\Delta_N L_{n,k+1}(\binom{n-1}{k} + p')|$$
$$= |\Delta_N L_{n-1,k}(\binom{n-1}{k})| + |\Delta_N L_{n-1,k+1}(p')|$$

(by Lemma 7.4)

$$\geq |\Delta_N L_{n-1,k}(\binom{n-1}{k})| + |\Delta_N L_{n-1,k}(p')|.$$  \hspace{1cm} (7.54)

(by Observation 2.7 and Theorem 2.48)

Also,

$$|\nabla_N L_{n,k-1}(p)| = |\nabla_N L_{n-1,k-2}(p')|$$

(by Lemma 2.31 as $p < \binom{n-1}{k-2}$)

$$= |\nabla_N L_{n-1,k-2}(p' + \binom{n-1}{k})|$$
$$= |\nabla_N L_{n-1,k-2}(p')| + |\nabla_N L_{n-1,k-2}(p' + \binom{n-1}{k})|.$$  \hspace{1cm} (by Observations 2.9 and 2.11)
Thus

\[
\left| \nabla_N L_{n,k-1}(p) \right| \geq \left| \nabla_N L_{n-1,k-2}(p') \right| + \left| \nabla_N F_{n-1,k-2}\left(\binom{n-1}{k}\right) \right|.
\]  
(7.55)

(by Note 2.52 and Corollary 2.49)

One concludes that

\[
\left| \triangle_N L_{n,k+1}(p) \right| + \left| \nabla_N L_{n,k-1}(p) \right| \\
\geq \left| \triangle_N F_{n-1,k}\left(\binom{n-1}{k}\right) \right| + \left| \triangle_N L_{n-1,k}(p') \right| \\
+ \left| \nabla_N L_{n-1,k-2}(p') \right| + \left| \nabla_N F_{n-1,k-2}\left(\binom{n-1}{k}\right) \right| \\
> 2\binom{n-1}{k} + 2p'
\]
(by Induction Hypothesis 7.15
as \(\binom{n-1}{k}, p' > 0\))

\[
= 2p.
\]

\[\square\]

4.2.2) \(\binom{n-2}{k-3} \leq \binom{n-1}{k}\)

This case is split into 2 parts.

4.2.2.1) \(\binom{n-1}{k} < p \leq \binom{n-1}{k-2}\)

Proposition 7.59. Let \(n > 32\) and \(k > \frac{n}{2}\) with \(\binom{n-2}{k-3} \leq \binom{n-1}{k}\). Then Theorem 7.1. (i) holds for \(\binom{n-1}{k} < p \leq \binom{n-1}{k-2}\).

Proof. The partitioning of \(L_{n,k+1}(p)\) and \(L_{n,k-1}(p)\) used in the proof is illustrated by Figure 7.22. Let \(p = p' + p'' + \binom{n-2}{k-3}\) be such that \(\binom{n-1}{k} = p'' + \binom{n-2}{k-3}\). By assumption
Figure 7.22: Partitioning of $L_{n,k+1}(p)$ and $L_{n,k-1}(p)$ in the proof of Proposition 7.59

$p > \binom{n-1}{k} \geq \binom{n-2}{k-3}$ so that $p' > 0$, $p'' \geq 0$, and $p' + p'' > 0$. For the new-shadow of $L_{n,k+1}(p)$ we write

$$\left| \Delta_N L_{n,k+1}(p) \right| = \left| \Delta_N L_{n,k+1} \left( p' + \binom{n-1}{k} \right) \right|$$

$$= \left| \Delta_N L_{n-1,k} \left( \binom{n-1}{k} \right) \right| + \left| \Delta_N L_{n-1,k+1} \left( p' \right) \right|$$

(by Lemma 7.4)

$$= \left| \Delta_N L_{n-1,k} \left( p'' + \binom{n-2}{k-3} \right) \right| + \left| \Delta_N L_{n-1,k+1} \left( p' \right) \right|$$

so that

$$\left| \Delta_N L_{n,k+1}(p) \right| = \left| \Delta_N L_{n-1,k} \left( \binom{n-2}{k-3} \right) \right| + \left| \Delta_N F_{n-1,k} \left( p'' \right) \right|$$

$$+ \left| \Delta_N L_{n-1,k+1} \left( p' \right) \right| .$$

(by Observation 2.10)

as $\binom{n-1}{k} = p'' + \binom{n-2}{k-3}$.
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For the new-shade of $L_{n,k-1}(p)$ we have

$$|\nabla N L_{n,k-1}(p)|$$

= $$|\nabla N L_{n-1,k-2}(p)|$$

(by Lemma 2.31

as $p \leq \binom{n-1}{k-2}$ by assumption)

= $$|\nabla N L_{n-1,k-2}(p') + p'' + \binom{n-2}{k-3}|$$

(by Lemma 7.4)

= $$|\nabla N L_{n-2,k-3}(p') + |\nabla N F_{n-2,k-2}(p'')|$$

(by Observations 2.9 and 2.11)

$$\geq |\nabla N L_{n-2,k-3}(p') + |\nabla N F_{n-2,k-2}(p'')|$$

(by Note 2.52 and Corollary 2.49)

so that

$$|\nabla N L_{n,k-1}(p)|$$

$$\geq |\nabla N L_{n-1,k-2}(p') + |\nabla N L_{n-1,k-1}(p') + |\nabla N F_{n-1,k-2}(p'')|.$$  \hspace{1cm} (7.5')

(by Lemmas 2.31 and 2.28)

We conclude that

$$|\triangle N L_{n,k+1}(p)| + |\nabla N L_{n,k-1}(p)|$$

$$\geq |\nabla N L_{n-1,k}(p') + |\nabla N F_{n-1,k}(p'')| + |\nabla N L_{n-1,k+1}(p')|$$

+ $$|\nabla N L_{n-1,k-2}(p') + |\nabla N L_{n-1,k-1}(p') + |\nabla N F_{n-1,k-2}(p'')|.$$  \hspace{1cm} (by (7.56) and (7.57))
That is,

\[
\left| \Delta_N L_{n, k+1} (p) \right| + \left| \nabla_N L_{n, k-1} (p) \right| > 2 \left( \binom{n-2}{k-3} \right) + 2 (p' + p'')
\]

(by Induction Hypothesis 7.15 as \( \binom{n-2}{k-3}, p' + p'' > 0 \))

\[
= 2p.
\]

\[\square\]

With the next case we reach the end of the proof of Theorem 7.1.(i).

4.2.2.2) \( \binom{n-1}{k-2} < p \leq \binom{n}{k+1} \)

**Proposition 7.60.** Let \( n > 32 \) and \( k > \frac{n}{2} \) with \( \binom{n-2}{k-3} \leq \binom{n-1}{k} \). Then Theorem 7.1.(i) holds for \( \binom{n-1}{k-2} < p \leq \binom{n}{k+1} \).

**Proof.** The partitioning of \( L_{n, k+1}(p) \) and \( L_{n, k-1}(p) \) used in the proof is illustrated by Figure 7.23. The proof assumes that Induction Hypothesis 7.15 holds and that Proposition 7.59 holds for \( n \).

Let \( p = p' + \binom{n-1}{k-2} \). As \( p > \binom{n-1}{k-2}, p' > 0 \). Also \( \binom{n-1}{k-2} \geq \binom{n-1}{k} \) for \( k > \frac{n}{2} \). First,

\[
\left| \Delta_N L_{n, k+1} (p) \right| = \left| \Delta_N L_{n, k+1} \left( p' + \binom{n-1}{k-2} \right) \right|
\geq \left| \Delta_N L_{n, k+1} \left( \binom{n-1}{k-2} \right) \right| + \left| \Delta_N L_{n-1, k+1} (p') \right|.
\]

(by Lemma 7.6 as \( \binom{n-1}{k-2} \geq \binom{n-1}{k} \))
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Figure 7.23: Partitioning of \( L_{n,k+1}(p) \) and \( L_{n,k-1}(p) \) in the proof of Proposition 7.60

Also,

\[
\left| \nabla_N L_{n,k-1}(p) \right| = \left| \nabla_N L_{n,k-1}\left(p' + \binom{n-1}{k-2}\right) \right| \\
= \left| \nabla_N L_{n-1,k-2}\left(\binom{n-1}{k-2}\right) \right| + \left| \nabla_N L_{n-1,k-1}(p') \right| \\
\quad \text{(by Lemma 7.4)} \\
= \left| \nabla_N L_{n,k-1}\left(\binom{n-1}{k-2}\right) \right| + \left| \nabla_N L_{n-1,k-1}(p') \right| . \quad (7.59)
\]

We conclude that

\[
\left| \Delta_N L_{n,k+1}(p) \right| + \left| \nabla_N L_{n,k-1}(p) \right| \\
\geq \left| \Delta_N L_{n,k+1}\left(\binom{n-1}{k-2}\right) \right| + \left| \Delta_N L_{n-1,k+1}(p') \right| \\
\quad + \left| \nabla_N L_{n,k-1}\left(\binom{n-1}{k-2}\right) \right| + \left| \nabla_N L_{n-1,k-1}(p') \right| . \\
\quad \text{(by (7.58) and (7.59))}
\]
That is,

\[ |\Delta N L_{n,k+1} (p)| + |\nabla_N L_{n,k-1} (p)| \]
\[ > 2 \left( \binom{n-1}{k-2} \right) + 2p' \]
\[ = 2p. \]

(by Proposition 7.59 applied to \( \binom{n-1}{k-2} \),
and Induction Hypothesis 7.15 as \( p' > 0 \))

This last proof concludes the proof of Theorem 7.1.(i).

### 7.7 Conclusion

The proof of Theorem 7.1 has been long but to date no other proof is known. It would be interesting to derive other inequalities by using a similar induction argument. One such inequality which is believed to hold is the following. For \( n, k, p, i \in \mathbb{Z}^+, \)
\[ i \leq k \leq n - i, \ 1 \leq p \leq \min \left\{ \binom{n}{k+1}, \binom{n}{k-1} \right\}, \]
\[ |\Delta N L_{n,k+1} (p)| + |\nabla_N L_{n,k-1} (p)| > 2p. \]

So far we have been unable to prove this inequality by using the method of proof of Theorem 7.1 due to the highly complex case discussion involved. This is further discussed in Chapter 9 (see Conjecture 9.3).

It is shown in Theorems 2.39 and 2.56 that \( |\Delta F_{n,k+1} (p)| \) and \( |\nabla_N F_{n,k-1} (p)| \) can be directly computed from the \((k+1)\)-binomial and the \((k-1)\)-binomial representation of \( p \) respectively. The \((k+1)\) and \((k-1)\)-binomial representations of \( p \) are independent of \( n \) (see Corollary 8.22). This suggests that to prove Theorem 7.1 an induction on \( p \) would be appropriate. We investigated this approach without success as a relationship between the \((k+1)\) and \((k-1)\)-binomial representations of \( p \) could not be found. Another approach taken was to consider the real binomial \( k \)-representation of \( p \). It was
thought that the real binomial representation of $p$ would be more convenient to use in investigating sizes of new-shadows and new-shades. While computing $|\Delta F_{n,k+1}(p)|$ can be simplified by using this approach (see Theorem 2.45), a suitable expression for $|\nabla N F_{n,k-1}(p)|$ could not be found.

Algorithm D.1 is used to prove that Theorem 7.1.(i) holds for values of $n$ less than 33 (see Proposition 7.14). As shown by the output of the program implemented from Algorithm D.1 (see Page 221) it is suspected that for $n$ fixed the function $\frac{|\nabla_N F_{n,k-1}(p)|}{p} + \frac{|\nabla_N F_{n,k+1}(p)|}{p}$ attains its minimum over all $k$ and $p$ when $k = \frac{n}{2} - 1$, $n$ even, and when $k = \frac{n-1}{2}$, $n$ odd, and $n$ sufficiently large. A strategy to prove Theorem 7.1 would then be to prove that this is indeed the case, and to prove that Theorem 7.1 holds for $k = \frac{n}{2} - 1$, $n$ even, and for $k = \frac{n-1}{2}$, $n$ odd, and $n$ sufficiently large. The latter seems reasonably easy to show, however the former appears difficult to prove.

These failed attempts explain why the proof of Theorem 7.1 has the present form since it turned out to be the only means by which proving the theorem was possible. It is therefore of interest, in the future, to look for a simplified proof, especially one using induction on $p$.

Major consequences of Theorem 7.1 are presented in Chapter 8 and include six special cases where the flat antichain conjecture is proven to hold.
Chapter 8

Some Consequences of the 3-Levels Result
8.1 Introduction

In this chapter several consequences of Theorem 7.1 are explored. In particular, we show that the flat antichain conjecture holds in six special cases. This is stated in Theorems 8.2, 8.7, 8.14, and 8.19, and Corollaries 8.4 and 8.5 respectively. Theorems 8.2 and 8.7 and Corollaries 8.4 and 8.5 are presented in Section 8.2, Theorems 8.14 and 8.19 are presented in Sections 8.4 and 8.5 respectively.

Theorem 8.2 says that the flat antichain conjecture holds for antichains \( \mathcal{A} \) with \( |A| \geq |\overline{A}| \) for each \( A \in \mathcal{A} \). There are two corollaries of Theorem 8.2. The first corollary is Corollary 8.4 which says that the flat antichain conjecture holds for antichains \( \mathcal{A} \) with \( |A| \leq |\overline{A}| \) for each \( A \in \mathcal{A} \). The second corollary is Corollary 8.5 which states that the flat antichain conjecture holds for antichains having sets on three consecutive levels. We will see that Corollary 8.5 can also be derived from Theorem 7.1 directly. Another consequence of Theorem 8.2 is Theorem 8.7 which shows that the flat antichain conjecture holds for antichains having \( \mathcal{A} \) with \( \overline{A} \leq 4 \).

Let \( \mathcal{A} \) be an antichain on \([n]\) with \( |\overline{A}| = k \) and with parameters \( p_i \). Let \( h \) be the largest integer for which \( p_i \neq 0 \) and assume that \( \sum_{i=1}^{h-k} ip_{k+i} \leq \binom{n}{k+1} \). Theorem 8.14 shows that the flat antichain conjecture holds for such an antichain. Therefore, to prove the truth of the flat antichain conjecture, it is left to investigate those antichains for which \( \sum_{i=1}^{h-k} ip_{k+i} > \binom{n}{k+1} \). This is done in Theorem 8.19 for antichains on four consecutive levels.

There are two other outcomes of Theorem 7.1 which are presented in this chapter, they are Theorem 8.10 and Corollary 8.22. Theorem 8.10 in Section 8.3 is one generalisation of Theorem 7.1(ii) and is easily obtained by the preliminary results needed to prove Theorem 8.2. In addition, Theorem 8.10 is a requirement to prove Theorem 8.14. Corollary 8.22 in Section 8.6 recasts Theorem 7.1(ii) as a new binomial inequality. Most results and proofs in Sections 8.2, 8.3, 8.4 and 8.5 arise from joint work with A. Woods.
8.2 The Flat Antichain Conjecture Holds for Antichains

\[ \mathcal{A} \text{ with } |A| \geq |\overline{A}| \text{ for each } A \in \mathcal{A} \]

We begin with the following note.

Note 8.1.

In this section and Section 8.3, no mention will be made of the universal set when considering collections \( F_k(p) \). The universal set is irrelevant in the context of Sections 8.2 and 8.3, since for given \( k \), only the number of the first \( k \)-sets in squashed order is of interest.

The main result in this section is Theorem 8.2 below. It is proved by applying Theorem 8.3 whose proof is deferred until the end of the section. Once Theorem 8.2 is proved, we state and prove two corollaries, Corollaries 8.4 and 8.5, and Theorem 8.7. Each of these three results make use of Theorem 8.2 to show that the flat antichain conjecture holds in certain special cases.

**Theorem 8.2.** Let \( \mathcal{A} \) be an antichain on \([n]\) with \(|A| \geq |\overline{A}|\) for each \( A \in \mathcal{A} \). Then the flat antichain conjecture holds for \( \mathcal{A} \).

A result significant in its own right and which is needed to prove Theorem 8.2 is Theorem 8.3.

**Theorem 8.3 (with Woods).** Let \( \mathcal{A} \) be a non-flat antichain on \([n]\) with \(|A| \geq |\overline{A}|\) for each \( A \in \mathcal{A} \). Let \( \mathcal{A}^* \) be the squashed flat counterpart of \( \mathcal{A} \). Then

\[ |\Diamond(|\overline{A}|)_A| > |\Diamond(|\overline{A}|)_{\mathcal{A}^*}|. \]

The proof of Theorem 8.3 requires Theorem 8.8 and one of its corollaries, Corollary 8.9. They are stated and proved towards the end of this section. The proof of Theorem 8.3 can be found after the proofs of Theorem 8.8 and Corollary 8.9. We prove Theorem 8.2 first.

**Proof of Theorem 8.2.** Assume that Theorem 8.3 holds. Let \( \mathcal{A} \) be an antichain on \([n]\) with \(|A| \geq |\overline{A}|\) for each \( A \in \mathcal{A} \). If \( \mathcal{A} \) is already flat then there is nothing to
prove. Assume that $A$ is not flat and let $A^*$ be the squashed flat counterpart of $A$.
Note that $A^*$ is an antichain. Further, $A^*$ is an antichain on $[n]$ by Lemma 2.79 since $|\triangle_l(A)| > |\triangle_l(A^*)|$ by Theorem 8.3. This proves the theorem.

Using Theorem 8.2 we prove that the flat antichain conjecture holds in three more special cases. This is stated in Corollaries 8.4, 8.5 and Theorem 8.7.

**Corollary 8.4.** Let $A$ be an antichain on $[n]$ with $|A| \leq |A|$ for each $A \in A$. Then the flat antichain conjecture holds for $A$.

**Proof.** Let $A$ be as in the statement of the corollary. Then the complement $A'$ of $A$ is an antichain with $A'$ such that $|A'| = n - |A|$ by Observation 2.82. Moreover, $|A| \geq n - |A|$ for each $A \in A'$. Therefore the flat antichain conjecture holds for $A'$ by Theorem 8.2, and for $A$ by Observation 4.4.

**Corollary 8.5.** Let $A$ be an antichain on $[n]$ with parameters $p$, and let $h$ and $l$ respectively be the largest and smallest integer for which $p_i \neq 0$. Assume that $h = k + 1$ and $l = k - 1$ for some $k \in \mathbb{Z}^+$. Then the flat antichain conjecture holds for $A$.

**Proof.** Let $A$ be as in the statement of the corollary. If $A = k$ then the flat antichain conjecture holds by Theorem 4.2. If $|A| = k - 1$ then the flat antichain conjecture holds by Theorem 8.2. If $|A| = k + 1$ then the flat antichain conjecture holds by Corollary 8.4.

Corollary 8.5 has an alternative proof which uses Theorem 7.1.

**Alternative proof of Corollary 8.5.** Let $A$ be as in the statement of the corollary. By Theorem 2.78 $A$ can be assumed to be squashed. Assume that $p_{k+1} \geq p_k$ and let $B = L(p_{k+1}, A^{(k+1)})$. Applying Corollary 2.37 and Theorems 2.47 and 7.1(i) together with Observation 2.9 we conclude that $|\triangle_N B| + |\nabla A^{(k+1)}| \geq |\triangle_N L_{k+1}(p)| + |\nabla L_{k-1}(p)| > 2p_{k-1}$. Thus there exists flat antichain on $[n]$ consisting of $p_{k+1} - p_{k-1}$ $(k + 1)$-sets and $p_k + 2p_{k-1}$ $k$-sets. The case $p_{k+1} < p_{k-1}$ is dealt with in a similar manner. This proves Corollary 8.5.
Chapter 8. Some Consequences of the 3-Levels Result

Before stating Theorem 8.7 we need the following result.

**Theorem 8.6 (Roberts [27]).** Let $A$ be an antichain on $[n]$ with $\overline{A} > 3$. Then the flat antichain conjecture holds for $A$ if it holds for all antichains $B$ on $[n]$ with $\overline{B} = \overline{A}$ and with $|B| \geq 3$ for each $B \in B$.

**Theorem 8.7.** Let $A$ be an antichain on $[n]$ with $\overline{A} \leq 4$. Then the flat antichain conjecture holds for $A$.

**Proof.** Let $A$ be as in the statement of the theorem. If $\overline{A} = 4$ then the flat antichain conjecture holds for $A$ by Theorem 4.2. If $\overline{A} \leq 3$ then the flat antichain conjecture holds for $A$ by Theorem 4.3. Therefore assume that $3 < \overline{A} < 4$ so that $|\overline{A}| = 3$. Then the flat antichain conjecture holds for all antichains $B$ on $[n]$ with $|B| \geq |\overline{B}| = |\overline{A}| = 3$ for each $B \in B$ by Theorem 8.2. It follows that the conjecture holds for $A$ by Theorem 8.6. \qed

We now state Theorem 8.8 which is the result leading to Theorem 8.3.

**Theorem 8.8 (with Woods).** Let $p_{k+1}, p_{k+2}, \ldots, p_{k+g}$ be non-negative integers. Then, for each $k$, $g \in \mathbb{Z}^+$, $g > 1$,

$$
|\Delta F_{k+1}(p_{k+1} + |\Delta F_{k+2}(p_{k+2} + |\Delta F_{k+3}(p_{k+3} + \ldots + |\Delta F_{k+g}(p_{k+g})|)|)|| + 
p_{k+2} + 2p_{k+3} + \ldots + (g - 1)p_{k+g} 
\geq |\Delta F_{k+1}(p_{k+1} + 2p_{k+2} + 3p_{k+3} + \ldots + gp_{k+g})|.
$$

Equality holds if and only if $p_{k+i} = 0$ for each $i = 2, \ldots, g$.

Before proving Theorem 8.8 we give two example applications of the theorem in order to highlight its significance. In both examples, assume that $A$ is a squashed antichain on $[n]$ with parameters $p_i$, that $\overline{A} = k + 1$, $k \in \mathbb{Z}^+$, and that $A$ is not flat. In the first example, assume that $A$ has sets on levels $k + 2$ and $k + 1$ only. That is, $p_k = p_{k+2}$. Assume that Theorem 8.8 holds. Then $|\Delta F_{k+1}(|\Delta F_{k+2}(p_{k+2})|)| + p_{k+2} = |\Delta^{(k)} F_{k+2}(p_{k+2})| + p_k > |\Delta F_{k+1}(2p_{k+2})|$. 


By Theorem 2.77, $|\Delta^{(k)} F_{k+2}(p_{k+2})| + p_k \leq \binom{n}{k}$ since $\mathcal{A}$ is an antichain on $[n]$. This implies that $|\Delta F_{k+1}(2p_{k+2})| < \binom{n}{k}$ so $2p_{k+2} < \binom{n}{k+1}$ as $|\Delta F_{k+1}\left(\binom{n}{k+1}\right)| = \binom{n}{k}$ by Observation 2.8. It follows that $F_{k+1}(2p_{k+2})$ is an antichain on $[n]$. As $F_{k+1}(2p_{k+2})$ is the squashed flat counterpart of $\mathcal{A}$, this result is a confirmation of Theorem 4.2.

Let $h$ be the largest $i$ for which $p_i \neq 0$. For the second example, assume that the sets of $\mathcal{A}$ are of size at least $k$ and that $h \geq k + 2$. If $\mathcal{A}$ has $p_{k+i}$ ($k + i$)-sets for $i = 1, \ldots, g$, then $\mathcal{A}$ has $p_k = \sum_{i=1}^{h-k} (i - 1)p_{k+i}$ sets on level $k$ as $|\mathcal{A}| = k + 1$. It follows that $|\mathcal{A}| = \sum_{i=1}^{h-k} i p_{k+i}$. Assuming that Theorem 8.8 holds, we have that $|\Delta^{(k)}\mathcal{A}| + p_k > |\Delta F_{k+1}(\{\mathcal{A}\})|$. Since $\mathcal{A}$ is an antichain on $[n]$ we have $|\Delta^{(k)}\mathcal{A}| + p_k \leq \binom{n}{k}$ and so $|\Delta F_{k+1}(\{\mathcal{A}\})| < \binom{n}{k}$. Therefore $F_{k+1}(\{\mathcal{A}\})$ is an antichain on $[n]$. Again, this confirms Theorem 4.2.

We now turn to the proof of Theorem 8.8. The main arguments used to prove Theorem 8.8 are Lemma 2.71 and Theorem 7.1. The thrust of the two arguments is the relationship which can be established between $|\Delta F_k(p)|$ and $|\nabla_N F_k(p)|$ for some appropriately chosen $h, l$ and $p$. Another argument used in the proof of Theorem 8.8 is Corollary 3.5. The reader may note how Corollary 3.5 enables the establishment of the strict inequality in Theorem 8.8 when $\sum_{i=2}^{g} p_{k+i} \neq 0$.

**Proof of Theorem 8.8.** The proof is by induction on $g$. Let the integers $p_{k+i}$, $1 \leq i \leq g$, be as in the statement of the theorem. Assume that $k, g \in \mathbb{Z}^+$, $g > 1$. We show that Theorem 8.8 holds for $g = 2$ and $k \geq 1$. Note that Theorem 8.8 trivially holds when $p_{k+2} = 0$, $g = 2$, and $k \geq 1$. Assume therefore that $p_{k+2} \neq 0$. Two cases are considered.

(i) Assume that $|\nabla_N F_k(p_{k+2})| = 0$. Then

$$
|\Delta F_{k+1}\left[p_{k+1} + |\Delta F_{k+2}(p_{k+2})|\right] + p_{k+2} > |\Delta F_{k+1}\left[p_{k+1} + |\Delta F_{k+2}(p_{k+2})| + |\nabla_N F_k(p_{k+2})|\right]|$$

$$(8.1)$$

(as $p_{k+2} \neq 0$ and $|\nabla_N F_k(p_{k+2})| = 0$)

$$
\geq |\Delta F_{k+1}\left[p_{k+1} + 2p_{k+2}\right]|.$$  

$$(8.2)$$

(by Observation 2.9 and Theorem 7.1 (ii))
(ii) Assume that $|\nabla N F_k(p_{k+2})| \neq 0$. Note that

$$|\Delta F_{k+2}(p_{k+2})| > 0$$

for all $p_{k+2} > 0$.

We have

$$|\Delta F_{k+1}[p_{k+1} + |\Delta F_{k+2}(p_{k+2})|] + p_{k+2}$$

$$\geq |\Delta F_{k+1}[p_{k+1} + |\Delta F_{k+2}(p_{k+2})|] + |\Delta F_{k+1}[\nabla N F_k(p_{k+2})]|$$

(by Lemma 2.71)

$$> |\Delta F_{k+1}[p_{k+1} + |\Delta F_{k+2}(p_{k+2})| + |\nabla N F_k(p_{k+2})|]$$

(by Corollary 3.5 and (8.3)

and the fact that $|\nabla N F_k(p_{k+2})| \neq 0$)

$$\geq |\Delta F_{k+1}[p_{k+1} + 2p_{k+2}]|$$

(by Observation 2.9 and Theorem 7.1(ii))

In both cases we conclude that

$$|\Delta F_{k+1}[p_{k+1} + |\Delta F_{k+2}(p_{k+2})|] + p_{k+2} > |\Delta F_{k+1}[p_{k+1} + 2p_{k+2}]|$$

by (8.2) and (8.5).

This proves that Theorem 8.8 holds for $g = 2$ and $k \geq 1$. Now assume that the theorem holds for $g - 1$ and for all $k \geq 1$. In particular, assume that it holds for $g - 1$ and $k + 1$. Then, the induction hypothesis implies that

$$|\Delta F_{(k+1)+1}[(p_{(k+1)+1}) + |\Delta F_{(k+1)+2}(p_{(k+1)+2}) + \ldots$$

$$+ |\Delta F_{(k+1)+(g-1)}(p_{(k+1)+(g-1)})]| + p_{(k+1)+2} + 2p_{(k+1)+3} + \ldots + (g - 2)p_{(k+1)+(g-1)}$$

$$- |\Delta F_{(k+1)+1}(p_{(k+1)+1}) + 2p_{(k+1)+2} + \ldots + (g - 1)p_{(k+1)+(g-1)}|$$

$$\geq 0.$$
This is equivalent to

\[
\| \Delta F_{k+2} \left[ p_{k+2} + \Delta F_{k+3}(p_{k+3} + \cdots + \Delta F_{k+2}(p_{k+2})) \right] \| + p_{k+3} + 2p_{k+4} + \cdots + (g - 2)p_{k+g} \\
- \| \Delta F_{k+2}(p_{k+2} + 2p_{k+3} + \cdots + (g - 1)p_{k+g}) \| \geq 0.
\]  
(8.6)

We show that Theorem 8.8 holds for \( g \) and all \( k \geq 1 \). Theorem 8.8 trivially holds if \( p_{k+i} = 0 \) for each \( i = 2, \ldots, g \). Therefore assume that \( p_{k+i} \neq 0 \) for some \( i = 2, \ldots, g \). In this case, we only need to show that

\[
\| \Delta F_{k+1} \left[ p_{k+1} + \Delta F_{k+2}(p_{k+2} + \Delta F_{k+3}(p_{k+3} + \cdots + \Delta F_{k+2}(p_{k+2})) \right] \| + p_{k+2} + 2p_{k+3} + \cdots + (g - 1)p_{k+g} \\
> \| \Delta F_{k+1} \left[ p_{k+1} + \Delta F_{k+2}(p_{k+2} + \Delta F_{k+3}(p_{k+3} + \cdots + \Delta F_{k+2}(p_{k+2})) \right] \| + \| \nabla N F_k(p_{k+2} + 2p_{k+3} + \cdots + (g - 1)p_{k+g}) \|.
\]  
(8.7)

Assume that (8.7) holds. Then,

\[
\| \Delta F_{k+1} \left[ p_{k+1} + \Delta F_{k+2}(p_{k+2} + \Delta F_{k+3}(p_{k+3} + \cdots + \Delta F_{k+2}(p_{k+2})) \right] \| + p_{k+2} + 2p_{k+3} + \cdots + (g - 1)p_{k+g} \\
> \| \Delta F_{k+1} \left[ p_{k+1} + \Delta F_{k+2}(p_{k+2} + \Delta F_{k+3}(p_{k+3} + \cdots + \Delta F_{k+2}(p_{k+2})) \right] \| + \| \nabla N F_k(p_{k+2} + 2p_{k+3} + \cdots + (g - 1)p_{k+g}) \| \tag{by (8.7)}
\]

\[
\geq \| \Delta F_{k+1} \left[ p_{k+1} + \Delta F_{k+2}(p_{k+2} + \Delta F_{k+3}(p_{k+3} + \cdots + \Delta F_{k+2}(p_{k+2})) \right] \| + 2p_{k+2} + 4p_{k+3} + \cdots + 2(g - 1)p_{k+g} \\
- \| \Delta F_{k+2}(p_{k+2} + 2p_{k+3} + \cdots + (g - 1)p_{k+g}) \| \tag{by Theorem 7.1.(ii)}
\]

\[
= \| \Delta F_{k+1} \left[ p_{k+1} + 2p_{k+2} + 3p_{k+3} + \cdots + gp_{k+g} \\
+ \Delta F_{k+2}(p_{k+2} + \Delta F_{k+3}(p_{k+3} + \cdots + \Delta F_{k+2}(p_{k+2})) \right] \| + p_{k+3} + 2p_{k+4} + \cdots + (g - 2)p_{k+g} \\
- \| \Delta N F_k(p_{k+2} + 2p_{k+3} + \cdots + (g - 1)p_{k+g}) \|.
\]
By (8.6) (the induction hypothesis), it follows that

\[
|\Delta F_{k+1}[p_{k+1} + |\Delta F_{k+2}(p_{k+2} + |\Delta F_{k+3}(p_{k+3} + \ldots + |\Delta F_{k+g}(p_{k+g})|)|)|]|
\]
\[
+ p_{k+2} + 2p_{k+3} + \ldots + (g - 1)p_{k+g}
\]
\[
> |\Delta F_{k+1}[p_{k+1} + 2p_{k+2} + 3p_{k+3} + \ldots + gp_{k+g}]|.
\]

as required. Therefore (8.7) implies that Theorem 8.8 holds. It remains to prove that (8.7) holds. Two cases are considered.

(a) \(|\nabla N F_k(p_{k+2} + 2p_{k+3} + \ldots + (g - 1)p_{k+g})| = 0\), and

(b) \(|\nabla N F_k(p_{k+2} + 2p_{k+3} + \ldots + (g - 1)p_{k+g})| \neq 0\).

Recall that \(p_{k+i} = 0\) for each \(i = 2, \ldots, g\), so \(\sum_{i=2}^{g} (i - 1)p_{k+i} \neq 0\). This fact, together with the same arguments as those used in (i) and (ii) to respectively obtain (8.1) and (8.4), can be used to show that, in both Cases (a) and (b),

\[
|\Delta F_{k+1}[p_{k+1} + |\Delta F_{k+2}(p_{k+2} + |\Delta F_{k+3}(p_{k+3} + \ldots + |\Delta F_{k+g}(p_{k+g})|)|)|]|
\]
\[
+ p_{k+2} + 2p_{k+3} + \ldots + (g - 1)p_{k+g}
\]
\[
> |\Delta F_{k+1}[p_{k+1} + |\Delta F_{k+2}(p_{k+2} + |\Delta F_{k+3}(p_{k+3} + \ldots + |\Delta F_{k+g}(p_{k+g})|)|)|]|
\]
\[
+ |\nabla N F_k(p_{k+2} + 2p_{k+3} + \ldots + (g - 1)p_{k+g})| |.
\]

This concludes the proof of Theorem 8.8. 

Next we state a corollary of Theorem 8.8.

**Corollary 8.9.** Let \(A\) be an antichain on \([n]\) with parameters \(p_i\). Let \(h\) be the largest integer for which \(p_i \neq 0\). Then

\[
|\Delta^{(k)} A| \geq |\Delta F_{k+1}(p_{k+1} + 2p_{k+2} + 3p_{k+3} + \ldots + (h - k)p_{h})|
\]
\[
- p_{k+2} - 2p_{k+3} - \ldots - (h - k - 1)p_{h}.
\]

Equality holds if and only if \(p_{k+i} = 0\) for each \(i = 2, \ldots, h - k\).

**Proof.** \(|\Delta^{(k)} A| \geq |\Delta F_{k+1}[p_{k+1} + |\Delta F_{k+2}(p_{k+2} + |\Delta F_{k+3}(p_{k+3} + \ldots + |\Delta F_{k}(p_{i})|)|)|]|

by repeated applications of Theorem 2.36. The result follows from Theorem 8.8 by
replacing \( g \) by \( h - k \). By Note 8.1, the collection \( \Delta F_{k+1} (p_{k+1} + \ldots + (h - k)p_h) \) might not be a collection of subsets of \([n]\) and be a collection of subsets of \([n']\), \( n' > n \), instead.

We are now in a position to prove Theorem 8.3.

**Proof of Theorem 8.3.** Let \( \mathcal{A} \) and \( \mathcal{A}^* \) be as in the statement of the theorem and let \( k = [\mathcal{A}] \). Assume that \( \mathcal{A} \) and \( \mathcal{A}^* \) have parameters \( p_i \) and \( q_i \) respectively. Let \( h \) be the largest integer for which \( p_i \neq 0 \). Since \( \mathcal{A} \) is not flat we have

\[
p_{k+i} \neq 0 \text{ for some } i = 2, \ldots, h - k. \tag{8.8}
\]

\( \mathcal{A}^* \) is a flat counterpart of \( \mathcal{A} \) by assumption. Thus, by Lemma 2.81, the parameters of \( \mathcal{A}^* \) are

\[
q_{k+1} = p_{k+1} + 2p_{k+2} + \ldots + (h - k)p_h \quad \text{and} \quad q_k = p_k - p_{k+2} - 2p_{k+3} - \ldots -(h - k - 1)p_h
\]

since \( p_{k+i} = 0 \) for \( i \geq 1 \). Now,

\[
|\Diamond^{(k)} \mathcal{A}| = |\Delta^{(k)} \mathcal{A}| + |\mathcal{A}^{(k)}| + |\nabla^{(k)} \mathcal{A}|
\]

(as \( \mathcal{A} \) is an antichain)

\[
> |\Delta F_{k+1} \left[ p_{k+1} + 2p_{k+2} + \ldots + (h - k)p_h \right]|
- p_{k+2} - 2p_{k+3} - \ldots -(h - k - 1)p_h
+ p_k
\]

(by Corollary 8.9 and (8.8),

and the fact that \( |\nabla^{(k)} \mathcal{A}| = 0 \))

\[
= |\Delta F_{k+1} (q_{k+1})| + q_k
\]

(by Lemma 2.81)

so that

\[
|\Diamond^{(k)} \mathcal{A}| > |\Diamond^{(k)} \mathcal{A}^*|
\]

as required.
8.3 One Generalisation of Theorem 7.1

There are several ways of generalising Theorem 7.1. One generalisation is Theorem 8.10 below. Other generalisations are stated in Chapter 9 as conjectures (see Conjecture 9.1 for example).

**Theorem 8.10.** Let $p_{k+1}, p_{k+2}, \ldots, p_{k+g}$ be non-negative integers. Then, for each $k, g \in \mathbb{Z}^+$,

$$|\Delta F_{k+1} [p_{k+1} + |\Delta F_{k+2} (p_{k+2} + |\Delta F_{k+3} (p_{k+3} + \ldots + |\Delta F_{k+g} (p_{k+g})))]]|$$

$$+ |\nabla N F_{k-1} (p_{k+1} + 2p_{k+2} + 3p_{k+3} + \ldots + gp_{k+g})|$$

$$\geq 2p_{k+1} + 3p_{k+2} + 4p_{k+3} + \ldots + (g + 1)p_{k+g}.$$  

Equality holds if and only if $p_{k+i} = 0$ for each $i = 1, \ldots, g$.

That Theorem 8.10 is a generalisation of Theorem 7.1 is made clearer by the following corollary of Theorem 8.10 which we state and prove before proceeding to the proof of Theorem 8.10.

**Corollary 8.11.** Let $k, g \in \mathbb{Z}^+$ be such that $k \geq 1$. Then, for $p \geq 0$,

$$|\Delta^{(k)} F_{k+g} (p)| + |\nabla N F_{k-1} (gp)| \geq (g + 1)p.$$  

Equality holds if and only if $p = 0$.

**Proof.** Assume that Theorem 8.10 holds. In Theorem 8.10 set $p_{k+i} = 0$ for $i = 1, \ldots, g - 1$. The result follows. Note that setting $g = 1$ gives Theorem 7.1.(ii). \hfill \Box

Theorem 8.10 derives almost immediately from Theorem 8.8 as shown below. Note that, in the proof below, it is Theorem 7.1.(ii) which enables the establishment of the strict inequality in Theorem 8.10 when $\sum_{i=1}^{g} p_{k+i} \neq 0$. 


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Proof of Theorem 8.10. Let \( n, k, g \) be as in the statement of the theorem. If \( p_{k+i} = 0 \) for each \( i = 1, \ldots, g \), then Theorem 8.10 trivially holds. Thus assume that \( p_{k+i} \neq 0 \) for some \( i = 1, \ldots, g \), so that \( \sum_{i=1}^{g} i p_{k+i} \neq 0 \). Now,

\[
|\Delta F_{k+1} + |\Delta F_{k+2}(p_{k+2} + |\Delta F_{k+3}(p_{k+3} + \ldots + |\Delta F_{k+g}(p_{k+g})|)|)|]
+ p_{k+2} + 2p_{k+3} + \ldots + (g - 1)p_{k+g}
\geq |\Delta F_{k+1}(p_{k+1} + 2p_{k+2} + 3p_{k+3} + \ldots + gp_{k+g})|
\]

(by Theorem 8.8)

\[
> 2p_{k+1} + 4p_{k+2} + 6p_{k+3} + \ldots + 2gp_{k+g}
- |\nabla_{k} F(k+1)(p_{k+1} + 2p_{k+2} + 3p_{k+3} + \ldots + gp_{k+g})|.
\]

(by Theorem 7.1(ii) as \( \sum_{i=1}^{g} i p_{k+i} \neq 0 \))

Therefore

\[
|\Delta F_{k+1} + |\Delta F_{k+2}(p_{k+2} + |\Delta F_{k+3}(p_{k+3} + \ldots + |\Delta F_{k+g}(p_{k+g})|)|)|)
+ |\nabla_{k} F(k+1)(p_{k+1} + 2p_{k+2} + 3p_{k+3} + \ldots + gp_{k+g})|
> 2p_{k+1} + 3p_{k+2} + 4p_{k+3} + \ldots + (g + 1)p_{k+g}
\]

as required.

The next corollary of Theorem 8.10 shows that the \( \overline{A} \)-image of a certain squashed antichain with \( \overline{A} \) an integer is uniquely minimised when \( A \) consists of \( |A| \overline{A} \)-sets.

Corollary 8.12. Let \( A \) be a squashed and non-flat antichain on \( [n] \) with \( \overline{A} \) an integer and \( |A| \geq \overline{A} - 1 \) for each \( A \subseteq A \). Then

\[
|\circ \overline{A} \{A\}| > |A|.
\]

Proof. Let \( \mathcal{A} \) be as in the statement of the theorem and let the integers \( p_i \) be the parameters of \( \mathcal{A} \) with \( h \) the largest integer for which \( p_i \neq 0 \). Let \( k = \overline{A} \), then \( p_{k-1} = \sum_{i=1}^{h-k} i p_{k+i} \). This implies that \( |A| = \sum_{i=0}^{h-k} (i + 1)p_{k+i} \). Then

\[
|\Delta^{(h-k)} A| + |\nabla_{k} F(k-1)| > \sum_{i=1}^{h-k} (i + 1)p_{k+i}
\]

as required.
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by Theorem 8.10. That is,

\[ |\Delta^{(k)} A| + p_k + |\nabla_N F_{k-1}(p_{k-1})| > \sum_{i=0}^{h-k} (i+1)p_{k+i} = |A|. \]

As \( A \) is a squashed antichain, \( |\phi(\overline{A}) A| = |\Delta^{(k)} A| + p_k + |\nabla_N A| = |\Delta^{(k)} A| + p_k + |\nabla_N A^{(k-1)}| \). Note that \( A^{(k-1)} \) is a collection \( C_{k-1}(p_{k-1}) \). The result then follows since \( |\nabla_N C_{k-1}(p_{k-1})| \geq |\nabla_N F_{k-1}(p_{k-1})| \) by Corollary 2.49.

For the class of antichains described by Corollary 8.12 it can be seen that Theorem 4.2 can be derived from Corollary 8.12 by applying Lemma 2.80 and noting that \( |A| = |\phi(\overline{A}) A^*| \) where \( A^* \) is any collection of \( |A| \ \overline{A} \)-sets.

8.4 Another Special Case of the Flat Antichain Conjecture

First, an introductory note.

Note 8.13.

In this section and in Section 8.5, we use the notation \( F_{n,k}, L_{n,k}, \ldots \), instead of the more concise \( F_{k}, L_{k}, \ldots \). This is needed in order to explicitly state the universal set \([n] \) where the collections of sets are defined.

There is another class of antichains for which the flat antichain conjecture can be shown to hold.

Theorem 8.14 (with Woods). Let \( A \) be an antichain on \([n] \) with parameters \( p_i \). Let \( h \) be the largest integer for which \( p_i \neq 0 \). Let \( |\overline{A}| = k \) and assume that \( \sum_{i=1}^{h-k} ip_{k+i} \leq {n \choose k+1} \). Then the flat antichain conjecture holds for \( A \).

The proof of Theorem 8.14 is given after stating Theorem 8.15 which is the major result required to prove Theorem 8.14. Note that Theorem 8.15 is similar to Theorem 8.3 but that it is stated for a different class of antichains.
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Theorem 8.15 (with Woods). Let \( A \) be a non-flat antichain on \([n]\) with parameters \( p_i \). Let \( h \) be the largest integer for which \( p_i \neq 0 \). Let \( |A| = k \) and assume that \( \sum_{i=1}^{h-k} ip_{k+i} \leq \binom{n}{k+1} \). Let \( A^* \) be the squashed flat counterpart of \( A \). Then

\[
|\phi(|A|)A| > |\phi(|A|)A^*|.
\]

Proving Theorem 8.15 requires proving three corollaries of previous theorems. The proof of Theorem 8.15 is deferred to the end of this section, after the proofs of the three corollaries. Using Theorem 8.15 we first prove Theorem 8.14.

Proof of Theorem 8.14. Assume that Theorem 8.15 holds. Let \( A \) be as in the statement of Theorem 8.14. If \( A \) is already flat then there is nothing to prove. So assume that \( A \) is not flat and let \( A^* \) be the squashed flat counterpart of \( A \). Note that \( A^* \) is an antichain. We show that \( A^* \) is an antichain on \([n]\). This is the case since \( |\phi(|A|)A| > |\phi(|A|)A^*| \) by Theorem 8.15, so \( A^* \) is an antichain on \([n]\) by Lemma 2.79.

Corollaries 8.16, 8.17, and 8.18 are needed to prove Theorem 8.15. Corollary 8.16 is a corollary of Theorem 8.8 while Corollaries 8.17 and 8.18 are corollaries of Theorem 8.10. We begin with Corollary 8.16 which is an alternative version of Corollary 8.9. The difference between Corollary 8.9 and Corollary 8.16 is that in the latter the universal set is explicitly stated. This is possible given the condition on the parameters of the antichain under consideration.

Corollary 8.16. Let \( A \) be an antichain on \([n]\) with parameters \( p_i \). Let \( h \) be the largest integer for which \( p_i \neq 0 \). Assume that \( \sum_{i=1}^{h-k} ip_{k+i} \leq \binom{n}{k+1} \). Then

\[
|\Delta^{(k)}A| \geq |\Delta F_{n,k+1}(p_{k+1} + 2p_{k+2} + 3p_{k+3} + \ldots + (h-k)p_k)|
- p_{k+2} - 2p_{k+3} - \ldots - (h-k-1)p_k.
\]

Equality holds if and only if \( p_{k+i} = 0 \) for each \( i = 2, \ldots, h-k \).

Proof. The assumption on \( \sum_{i=1}^{h-k} ip_{k+i} \) implies that the collection \( F_{n,k+1}(p_{k+1} + 2p_{k+2} + 3p_{k+3} + \ldots + (h-k)p_k) \) is defined. The result follows from Corollary 8.9.
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We now state and prove the two corollaries of Theorem 8.10.

**Corollary 8.17.** Let \( A \) be an antichain on \([n]\) with parameters \( p_i \). Let \( h \) be the largest integer for which \( p_i \neq 0 \). Assume that \( \sum_{i=1}^{h-1} iP_{k+i} \leq \binom{n}{k-1} \). Then

\[
|\Delta^{(k)} A| + |\nabla_N F_{n,k-1}(p_{k+1} + 2p_{k+2} + 3p_{k+3} + \ldots + (h - k)p_h)|
\geq 2p_{k+1} + 3p_{k+2} + 4p_{k+3} + \ldots + (h - k + 1)p_h.
\]

Equality holds if and only if \( p_{k+i} = 0 \) for each \( i = 1, \ldots, h - k \).

**Proof.** \( |\Delta^{(k)} A| \geq |\Delta F_{n,k+1}| \left[ p_{k+1} + |\Delta F_{k+2}(p_{k+2} + |\Delta F_{k+3}(p_{k+3} + \ldots + |\Delta F_h(p_h)|)|)| \right] \)
by repeated applications of Theorem 2.36. The condition \( \sum_{i=1}^{h-1} iP_{k+i} \leq \binom{n}{k-1} \) ensures that the collection \( \nabla_N F_{n,h-1}(p_{k+1} + 2p_{k+2} + 3p_{k+3} + \ldots + (h - k)p_h) \) is defined. The result follows from Theorem 8.10 by replacing \( g \) by \( h - k \). \( \square \)

**Corollary 8.18.** Let \( A \) be an antichain on \([n]\) with parameters \( p_i \). Let \( l \) be the smallest integer for which \( p_i \neq 0 \). Assume that \( \sum_{i=1}^{k-l} iP_{k-i} \leq \binom{n}{k+1} \). Then

\[
|\nabla^{(k)} A| + |\Delta_N E_{n+1}(p_{k-1} + 2p_{k-2} + 3p_{k-3} + \ldots + (k - l)p_l)|
\geq 2p_{k-1} + 3p_{k-2} + 4p_{k-3} + \ldots + (k - l)p_l.
\]

Equality holds if and only if \( p_{k-i} = 0 \) for each \( i = 1, \ldots, k - l \).

**Proof.** Let \( A \) be as in the statement of the lemma. If \( p_{k-i} = 0 \) for each \( i = 1, \ldots, k - l \), then \( \nabla^{(k)} A = \emptyset \) and the result follows. Therefore assume that

\[
p_{k-i} \neq 0 \text{ for some } i = 1, \ldots, k - l.
\]

(8.9)

Consider the complement \( A' \) of \( A \). Assume that \( A' \) has parameters \( q_i \). Then \( q_{n-i} = p_i \) and \( \sum_{i=1}^{k-l} iq_{n-k+i} = \sum_{i=1}^{k-l} iP_{k-i} \leq \binom{n}{k+1} = \binom{n}{n-k-1} \). Also, \( q_{n-k+i} \neq 0 \) for some \( i = 1, \ldots, k - l \) by (8.9). It follows that

\[
|\Delta^{(n-k)} A'| + |\nabla_N F_{n,n-k-1}(q_{n-k+1} + 2q_{n-k+2} + 3q_{n-k+3} + \ldots + (k - l)q_{n-i})|
> 2q_{n-k+1} + 3q_{n-k+2} + 4q_{n-k+3} + \ldots + (k - l + 1)q_{n-i}.
\]
Chapter 8. Some Consequences of the 3-Levels Result

by Corollary 8.17. Thus

$$|\nabla^{(k)} A| + |\Delta N_{n,k+1}(q_{n-k+1} + 2q_{n-k+2} + 3q_{n-k+3} + \ldots + (k-l)q_{n-l})|$$

$$> 2q_{n-k+1} + 3q_{n-k+2} + 4q_{n-k+3} + \ldots + (k-l+1)q_{n-l}.$$  

by repeated applications of Lemma 2.21 together with Observations 2.3 and 2.9. Therefore

$$|\nabla^{(k)} A| + |\Delta N_{n,k+1}(p_{k-1} + 2p_{k-2} + 3p_{k-3} + \ldots + (k-l)p_l)|$$

$$> 2p_{k-1} + 3p_{k-2} + 4p_{k-3} + \ldots + (k-l+1)p_l$$

as required. □

We are now ready to prove Theorem 8.15.

Proof of Theorem 8.15. Let $A$ and $A^*$ be as in the statement of the theorem. In particular, $|A| = k$ and

$$\sum_{i=1}^{h-k} i p_{k+i} \leq \binom{n}{k+1}. \quad (8.10)$$

Let $l$ be the smallest $i$ for which $p_i \neq 0$. Since $A$ is not flat and $|A| = k$,

$$p_{k+i} \neq 0 \text{ for some } i = 2, \ldots, h-k, \text{ or } p_{k-i} \neq 0 \text{ for some } i = 1, \ldots, k-l.$$

(8.11)

Assume that $A^*$ has parameters $q_i$. By Lemma 2.81, $q_{k+1} = \sum_{i=1}^{h-k} i p_{k+i} - \sum_{i=1}^{h-l} i p_{k-i}$ and $q_k = p_k + \sum_{i=1}^{k-l} (i+1)p_{k-i} - \sum_{i=2}^{h-k} (i-1)p_{k+i}$. Note that $q_{k+1} \geq 0$ as $|A^*| = |A| = k$. Therefore $\sum_{i=1}^{h-k} i p_{k+i} \geq \sum_{i=1}^{h-l} i p_{k-i}$ and

$$\sum_{i=1}^{k-l} i p_{k-i} \leq \binom{n}{k+1} \quad (8.12)$$
by (8.10). We show that $|\diamondsuit^{(k)} \mathcal{A}| > |\diamondsuit^{(k)} \mathcal{A}^*|$. We have

$$
|\diamondsuit^{(k)} \mathcal{A}| = |\triangle^{(k)} \mathcal{A}| + |\mathcal{A}^{(k)}| + |\nabla^{(k)} \mathcal{A}|
$$

(as $\mathcal{A}$ is an antichain)

$$
> |\triangle F_{n,k+1} (p_{k+1} + 2p_{k+2} + \ldots + (h - k)p_h) | \\
- p_{k+2} - 2p_{k+3} - \ldots - (h - k - 1)p_h \\
+ p_h \\
- |\triangle N L_{n,k+1} (p_{k-1} + 2p_{k-2} + 3p_{k-3} + \ldots + (k - l)p_l) | \\
+ 2p_{k-1} + 3p_{k-2} + 4p_{k-3} + \ldots + (k - l + 1)p_l
$$

(by Corollaries 8.16 and 8.18, and (8.10), (8.12), and (8.11))

$$
= |\triangle F_{n,k+1} \left( \sum_{i=1}^{h-k} ip_{k+i} \right) | - |\triangle N L_{n,k+1} \left( \sum_{i=1}^{k-l} ip_{l-i} \right) | \\
+ p_h + \sum_{i=1}^{k-l} (i + 1)p_{k-i} - \sum_{i=2}^{h-k} (i - 1)p_{k+i}.
$$

It follows that

$$
|\diamondsuit^{(k)} \mathcal{A}| > \left|\triangle F_{n,k+1} \left( \sum_{i=1}^{h-k} ip_{k+i} - \sum_{i=1}^{k-l} ip_{l-i} \right) \right| \\
+ p_h + \sum_{i=1}^{k-l} (i + 1)p_{k-i} - \sum_{i=2}^{h-k} (i - 1)p_{k+i}
$$

(by Corollary 2.54

as $\sum_{i=1}^{h-k} ip_{k+i} \geq \sum_{i=1}^{k-l} ip_{l-i}$)

$$
= |\triangle F_{n,k+1} (q_{k+1})| + q_h
$$

(by Lemma 2.81)

so that

$$
|\diamondsuit^{(k)} \mathcal{A}| > |\diamondsuit^{(k)} \mathcal{A}^*|
$$

as required. \qed
Let $\mathcal{A}$ be an antichain as in Theorems 8.14 or 8.15. Were the condition $\sum_{i=1}^{h-k} i p_{k+i} \leq \binom{n}{k+1}$ not required for the proof of Theorem 8.15 then the flat antichain conjecture would hold for all antichains. The next theorem, Theorem 8.19, shows that the conjecture holds for antichains on four consecutive levels. It will be seen that the proof of Theorem 8.19 requires a lengthy discussion in the case when the condition $\sum_{i=1}^{h-k} i p_{k+i} \leq \binom{n}{k+1}$ is not met.

### 8.5 The Flat Antichain Conjecture Holds for Antichains on 4 Consecutive Levels

Here, as in Section 8.4, the notation $F_{n,k}$, $L_{n,k}$, ... will be used instead of $F_k$, $L_k$, ... (see Note 8.13). The main result in this section is the following theorem which solves the flat antichain conjecture in another special case, namely when the antichain consists of sets on four consecutive levels.

**Theorem 8.19 (with Woods).** Let $\mathcal{A}$ be an antichain on $[n]$ with parameters $p_i$ and let $h$ and $l$ respectively be the largest and smallest integer for which $p_i \neq 0$. Assume that $h = k + 2$ and $l = k - 1$ for some $k \in \mathbb{Z}^+$. Then the flat antichain conjecture holds for $\mathcal{A}$.

A definition is required before proving Theorem 8.19.

**Definition 8.20.**

In the proof of Theorem 8.19 $n'$ is the smallest size universal set, but no smaller than $n$, for which the collections of consecutive sets in squashed order under consideration are defined.

**Proof of Theorem 8.19.** Let $\mathcal{A}$ be as in the statement of the theorem. Note that $p_{k+2}$, $p_{k-1} \neq 0$. It can be assumed that $p_{k+1} + 2p_{k+2} > \binom{n}{k+1}$ as if $p_{k+1} + 2p_{k+2} \leq \binom{n}{k+1}$ then the flat antichain conjecture holds for $\mathcal{A}$ by Theorem 8.14. If $\overline{A}$ is an integer then the flat antichain conjecture holds for $\mathcal{A}$ by Theorem 4.2. Thus assume that $\overline{A}$ is not an integer so $|\overline{A}| < \overline{A}$ and $|\overline{A}| > \overline{A}$. 
\( \lceil A \rceil \) can take three different values:

1. \( \lceil A \rceil = k - 1 \),
2. \( \lceil A \rceil = k \),
3. \( \lceil A \rceil = k + 1 \).

In Case (1), the flat antichain conjecture holds for \( A \) by Theorem 8.2. In Case (3), the flat antichain conjecture holds for \( A \) by Corollary 8.4 as \( \lceil A \rceil = k + 2 \). Thus assume that \( \lceil A \rceil = k \). If \( k > \frac{n-1}{2} \) then \( \lceil A' \rceil = n - k - 1 < \frac{n-1}{2} \) by Observation 2.82.

By Observation 4.4, if the flat antichain conjecture holds for \( A' \) then it holds for \( A \). We can therefore assume that \( k \leq \frac{n-1}{2} \). It follows that \( \binom{n}{k-1} \leq \binom{n}{k+1} \) and that the collection \( L_{n,k+1}(p_{k-1}) \) exists.

Let \( A^* \) be the squashed flat counterpart of \( A \). Assume that \( A^* \) has parameters \( q_i \). Then \( q_{k+1} = p_{k+1} + 2p_{k+2} - p_{k-1} \) and \( q_k = p_k + 2p_{k-1} - p_{k+2} \) by Lemma 2.81.

**Claim 1.** \( q_{k+1} \leq \binom{n}{k+1} \).

*Proof of the Claim.* The parameters of \( A^* \) satisfy the LYM inequality by Theorem 4.9. That is, \( \frac{\binom{k+1}{k+1}}{\binom{k+1}{k+1}} + \frac{\binom{k}{k}}{\binom{k}{k}} \leq 1 \). For \( k \leq \frac{n-1}{2} \), \( \binom{n}{k} \geq \binom{n}{k+1} \), so that \( \frac{\binom{k+1}{k+1}}{\binom{k+1}{k+1}} + \frac{\binom{k}{k}}{\binom{k}{k}} \leq \frac{\binom{k+1}{k+1}}{\binom{k+1}{k+1}} + \frac{\binom{k}{k}}{\binom{k}{k}} \leq 1 \). This proves the claim.

Let \( L = |\triangle F_{n,k+1}(p_{k+1})| + |\triangle N L_{n,k+1}(p_{k-1})| - p_{k+2} = |\triangle F_{n,k+1}(p_{k+1} + 2p_{k+2} - p_{k-1})| + |\triangle N L_{n,k+1}(p_{k-1})| - p_{k+2} \) and \( R = |\triangle F_{n,k+1}(p_{k+1} + |\triangle F_{n,k+2}(p_{k+2})|) \). Note that the collections of sets of which \( L \) consists of exist by Claim 1 and the assumption that \( k < \frac{n}{2} \).

**Claim 2.** Assume that \( L \leq R \). Then Theorem 8.19 holds.

*Proof of the Claim.* Assume that \( R \geq L \). Then,

\[
\begin{align*}
|\triangle F_{n,k+1}(p_{k+1} + |\triangle F_{n,k+2}(p_{k+2})|) & \\
\geq |\triangle F_{n,k+1}(p_{k+1} + 2p_{k+2} - p_{k-1})| + |\triangle N L_{n,k+1}(p_{k-1})| - p_{k+2} \\
> |\triangle F_{n,k+1}(p_{k+1} + 2p_{k+2} - p_{k-1})| + 2p_{k-1} - |\triangle N L_{n,k-1}(p_{k-1})| - p_{k+2}. \quad (8.13)
\end{align*}
\]

(by Theorem 7.1.(i) as \( p_{k-1} > 0 \))
Thus,

\[ |\Theta^{(k)} \mathcal{A}| = |\Delta^{(k)} \mathcal{A}| + |\mathcal{A}^{(k)}| + |\nabla^{(k)} \mathcal{A}| \]

(as \( \mathcal{A} \) is an antichain)

\[ \geq |\Delta F_{n,k+1} (p_{k+1} + |\Delta F_{n,k+2} (p_{k+2})|)| + p_k + |\nabla N L_{n,k-1} (p_{k-1})| \]

(by Theorem 2.36 and Corollary 2.37)

\[ > |\Delta F_{n,k+1} (p_{k+1} + 2p_{k+2} - p_{k-1})| + p_k + 2p_{k-1} - p_{k+2} \]

(by (8.13))

\[ = |\Delta F_{n,k+1} (q_{k+1})| + q_k. \]

Equivalently,

\[ |\Theta^{(k)} \mathcal{A}| > |\Theta^{(k)} \mathcal{A}^*|. \]

(8.14)

It follows that \( \mathcal{A}^* \) is an antichain on \([n]\) by Lemma 2.79. This proves the claim. It remains therefore to prove that \( L \leq R \). We consider two cases.

(i) Assume that \( p_{k+1} + 2p_{k+2} > (\binom{n}{k+1}) \) and \( 2p_{k+2} - |\Delta F_{n,k+2} (p_{k+2})| \leq (\binom{n}{k+1}) \). For convenience, we restate our assumptions and make several simple observations.

\[ p_{k+1} + 2p_{k+2} > (\binom{n}{k+1}) \]  \hspace{1cm} (O1)

\[ 2p_{k+2} - |\Delta F_{n,k+2} (p_{k+2})| \leq (\binom{n}{k+1}) \]  \hspace{1cm} (O2)

\[ (\binom{n}{k+1}) - p_{k-1} \geq 0 \]  \hspace{1cm} (O3)

since \( p_{k-1} \leq (\binom{n}{k-1}) \leq (\binom{n}{k+1}) \) and \( k < \frac{n}{2} \).

\[ p_{k+1} + 2p_{k+2} - (\binom{n}{k+1}) > 0 \]  \hspace{1cm} (O4)

by (O1).

\[ (\binom{n}{k+1}) + |\Delta F_{n,k+2} (p_{k+2})| - 2p_{k+2} \geq 0 \]  \hspace{1cm} (O5)

by (O2).

\[ p_{k+1} + |\Delta F_{n,k+2} (p_{k+2})| > (\binom{n}{k+1}) + |\Delta F_{n,k+2} (p_{k+2})| - 2p_{k+2} \]  \hspace{1cm} (O6)

by (O1).
Now,
\[
L = |\Delta F_{n,k+1}(p_{k+1} + 2p_{k+2} - p_{k-1})| + |\Delta N L_{n,k+1}(p_{k-1}) - p_{k+2}|
\]
\[
\leq |\Delta F_{n,k+1}\left(\binom{n}{k+1} - p_{k-1}\right)| + |\Delta F_{n,k+1}\left(p_{k+1} + 2p_{k+2} - \binom{n}{k+1}\right)| + |\Delta N L_{n,k+1}(p_{k-1}) - p_{k+2}|
\]
\[
\quad \text{(by Corollary 2.53 and (O3) and (O4))}
\]
\[
\leq |\Delta F_{n,k+1}\left(\binom{n}{k+1}\right)| - |\Delta N L_{n,k+1}(p_{k-1})|
\]
\[
\quad + |\Delta F_{n,k+1}\left(p_{k+1} + 2p_{k+2} - \binom{n}{k+1}\right)| + |\Delta N L_{n,k+1}(p_{k-1})| - p_{k+2}
\]
\[
\quad \text{(by Corollary 2.54 and (O3))}
\]
\[
= |\Delta F_{n,k+1}\left(\binom{n}{k+1}\right)| + |\Delta F_{n,k+1}\left(p_{k+1} + 2p_{k+2} - \binom{n}{k+1}\right)| - p_{k+2}
\]
\[
= \binom{n}{k} - p_{k+2} + |\Delta F_{n,k+1}\left(p_{k+1} + 2p_{k+2} - \binom{n}{k+1}\right)|.
\]
\[
\quad \text{(by Observation 2.8)}
\]
\[
\leq \binom{n}{k} - p_{k+2} + |\Delta F_{n,k+1}\left(p_{k+1} + |\Delta F_{n,k+1}(p_{k+2})|\right)|
\]
\[
\quad - |\Delta N L_{n,k+1}\left(\binom{n}{k+1} + |\Delta F_{n,k+1}(p_{k+1})| - 2p_{k+2}\right)|
\]
\[
\quad \text{(by Corollary 2.54 and (O5) and (O6))}
\]
\[
= R + \binom{n}{k} - p_{k+2} - |\Delta N L_{n,k+1}\left(\binom{n}{k+1} - (2p_{k+2} - |\Delta F_{n,k+1}(p_{k+1})|)\right)|
\]
\[
= R + \binom{n}{k} - p_{k+2} - |\Delta N F_{n,k+1}\left(\binom{n}{k+1}\right)|
\]
\[
\quad + |\Delta N F_{n,k+1}\left(2p_{k+2} - |\Delta F_{n,k+1}(p_{k+1})|\right)|
\]
\[
\quad \text{(by the definition of new-shadow and (O2))}
\]
\[
= R + \binom{n}{k} - p_{k+2} - \binom{n}{k} + |\Delta N F_{n,k+1}\left(2p_{k+2} - |\Delta F_{n,k+1}(p_{k+1})|\right)|
\]
\[
\quad \text{(by Observation 2.8)}
\]
\[
\leq R - p_{k+2} + |\Delta N F_{n',k+1}\left(\nabla N F_{n',k}(p_{k+2})\right)|
\]
\[
\quad \text{(by Theorem 7.1 (ii) using Definition 8.20)}
\]
\[
\leq R - p_{k+2} + p_{k+2}
\]
\[
\quad \text{(by Lemma 2.71 and Observation 2.9)}
\]
\[
= R.
\]
(ii) Assume that \( p_{k+1} + 2p_{k+2} > \binom{n}{k+1} \) and \( 2p_{k+2} - |\Delta F_{n,k+2}(p_{k+2})| > \binom{n}{k+1} \). Recall that \( q_{k+1} = p_{k+1} + 2p_{k+2} - p_{k-1} \leq \binom{n}{k+1} \) by Claim 1.

The second assumption says that \( q_{k+1} = p_{k+1} + 2p_{k+2} - p_{k-1} > p_{k+1} + \binom{n}{k+1} + |\Delta F_{n,k+2}(p_{k+2})| - p_{k-1} \). Hence \( \binom{n}{k+1} \geq q_{k+1} > |\Delta F_{n,k+2}(p_{k+2})| - p_{k-1} + \binom{n}{k+1} \). It follows that \( |\Delta F_{n,k+2}(p_{k+2})| - p_{k-1} < 0 \), that is, \( \binom{n}{k-1} \geq p_{k-1} > |\Delta F_{n,k+2}(p_{k+2})| \).

However, \( |\Delta F_{n,k+2}(p_{k+2})| \geq \frac{k+2}{n-k+1}p_{k+2} \) by Sperner’s lemma. Thus,

\[
\binom{n}{k-1} > \frac{k + 2}{n - k - 1}p_{k+2}. \tag{8.15}
\]

Further,

\[
|\nabla_{N} F_{n',k}(p_{k+2})| \geq 2p_{k+2} - |\Delta F_{n',k+2}(p_{k+2})| \geq 2p_{k+2} - \binom{n}{k+1} \tag{by Theorem 7.1(ii)}
\]

and using Definition 8.20

\[
= 2p_{k+2} - |\Delta F_{n,k+2}(p_{k+2})| \geq \binom{n}{k-1}. \tag{by Observation 2.6 as \( p_{k+2} \leq \binom{n}{k+1} \)}
\]

(by assumption)

It follows that

\[
p_{k+2} > \binom{n}{k}. \tag{8.16}
\]

since \( |\nabla_{N} F_{n',k}(\binom{n}{k})| = |\nabla_{N} F_{n,k}(\binom{n}{k+1})| = \binom{n}{k+1} \) by Observations 2.6 and 2.8.

Combining (8.15) and (8.16) gives

\[
\binom{n}{k-1} > \frac{k + 2}{n - k - 1}p_{k+2} > \frac{k + 2}{n - k - 1} \binom{n}{k} > \frac{k}{n - k + 1} \binom{n}{k} = \binom{n}{k-1}. \tag{by assumption}
\]

This contradiction solves Case (ii) and concludes the proof of Theorem 8.19. \( \square \)

**Corollary 8.21.** Let \( \mathcal{A} \) be a non-flat antichain on \([n]\) with parameters \( p_i \) and \( |\mathcal{A}| = k \). Let \( h \) and \( l \) respectively be the largest and smallest integer for which \( p_i \neq 0 \) and
assume that \( h = k + 2 \) and \( l = k - 1 \). Let \( A^* \) be the squashed flat counterpart of \( A \). Then

\[
|\diamondsuit([\mathcal{A}])_A| > |\diamondsuit([\mathcal{A}])_{A^*}|.
\]

\textit{Proof.} If \( \overline{A} \) is not an integer then \( A^* \) is as in the proof of Theorem 8.19 and the result follows from (8.14). If \( \overline{A} \) is an integer then \( |\mathcal{A}| = \overline{A} = k \) and the result follows from Corollary 8.12 and the fact that \( \diamondsuit(\mathcal{A})_A \supseteq \diamondsuit(\mathcal{A})_{A^*} = \diamondsuit(\overline{A})_{A^*} = F_{n,k}(|\mathcal{A}|) \).

\section{Theorem 7.1 & A New Binomial Inequality}

In this last section of the chapter we recast Theorem 7.1.(ii) in terms of a binomial inequality.

\textbf{Corollary 8.22.} Let \( n, k \in \mathbb{Z}^+ \) be such that \( 0 \leq k - 1 < k + 1 \leq n \) and let \( p \in \mathbb{Z}^+ \). Assume that the \((k + 1)\)-binomial representation of \( p \) is \( \sum_{i=0}^{k+1} \binom{a_i}{i} \) and that the \((k - 1)\)-binomial representation of \( p \) is \( \sum_{i=1}^{k-1} \binom{b_i}{i} \). Then

\[
\sum_{i=m}^{k+1} \binom{a_i}{i-1} + \sum_{i=m}^{k-1} \binom{b_i}{i+1} > 2p.
\]

\textit{Proof.} Let the \((k + 1)\) and \((k - 1)\)-binomial representations of \( p \) be as given in the statement of the corollary. By Theorem 2.39

\[
|\Delta F_{k+1}(p)| = \sum_{i=m}^{k+1} \binom{a_i}{i-1},
\]

and by Theorem 2.56,

\[
|\nabla_N F_{k-1}(p)| = \sum_{i=m}^{k-1} \binom{b_i}{i+1}.
\]

By Theorem 7.1.(ii), \( |\Delta F_{k+1}(p)| + |\nabla_N F_{k-1}(p)| > 2p \). The result then follows from (8.17) and (8.18). \( \square \)
Chapter 9

The 3-Levels Result and the Flat Antichain Conjecture:
Discussion and More Conjectures
Chapter 9. The 3-Levels Result and the Flat Antichain Conjecture: Discussion

9.1 Introduction

In this chapter we present some conjectures and open problems arising from Theorem 7.1 and the results in Chapter 8. In Section 9.2 several conjectures are stated which arose while proving Theorem 7.1. We mention Conjecture 9.1 which is another possible generalisation of Theorem 7.1. Conjecture 9.1 can be used to derive a similar result (Theorem 9.5) to that of Corollary 8.12. Under several assumptions, including the assumption that Conjecture 9.1 holds, Theorem 9.5 shows that the $\overline{A}$-image of a squashed and non-flat antichain $A$ with $\overline{A}$ an integer and $|A| \leq \overline{A} + 1$ for each $A \in A$ is uniquely minimised when $A$ consists of $\overline{A}$-sets only.

Section 9.3 presents Conjecture 9.7 which is a conjecture in relation to the size of the $\overline{A}$-image of a squashed antichain $A$ with $\overline{A}$ an integer. Conjecture 9.7 states that the size of the $\overline{A}$-image is uniquely minimised when $A$ consists of $\overline{A}$-sets only. Further, a relationship is established between Conjecture 9.7 and Theorem 5.1. This is done in Section 9.4.

Section 9.5 presents Conjecture 9.9 which is a conjecture in relation to the size of the $|A|$-projection of an antichain $A$. Conjecture 9.9 states that the size of the $|A|$-projection of $A$ is uniquely minimised when $A$ is a squashed and flat antichain.

Section 9.6 briefly discusses other possible approaches to solving the flat antichain conjecture. Section 9.7 concludes the chapter with an open problem which might help in proving the flat antichain conjecture.

9.2 Another Generalisation and Variants of Theorem 7.1

All the conjectures presented in this section arose while proving Theorem 7.1. In Chapter 8 we have seen that Theorem 7.1 can be used to show that the flat antichain conjecture holds in six cases. In doing so we were able to formulate one generalisation of Theorem 7.1 as given by Theorem 8.10. There is another possible generalisation of Theorem 7.1 which we state as a conjecture.
Chapter 9. The 3-Levels Result and the Flat Antichain Conjecture: Discussion

Conjecture 9.1. Let \( p_{k-1}, p_{k-2}, \ldots, p_{k-g} \) be non-negative integers. Then, for each \( k, g \in \mathbb{Z}^+ \),

\[
|\triangle F_{k+1}(p_{k-1} + 2p_{k-2} + 3p_{k-3} + \ldots + gp_{k-g})| \\
+ |\nabla_N F_{k-1}(p_{k-1} + \nabla_N F_{k-2}(p_{k-2} + \nabla_N F_{k-3}(p_{k-3} + \ldots \\
\quad + \nabla_N F_{k-g}(p_{k-g})))| |i|| \\
\geq 2p_{k-1} + 3p_{k-2} + 4p_{k-3} + \ldots + (g + 1)p_{k-g}.
\]

Equality holds if and only if \( p_{k-i} = 0 \) for each \( i = 1, \ldots, g \).

By setting all the \( p_{k-i} \)s to 0 for \( i = 2, \ldots, g \) we see that Theorem 7.1.(ii) is a special case of Conjecture 9.1. An application of the validity of Conjecture 9.1 will be given at the end of this section in Theorem 9.5. By setting all the \( p_{k-i} \)s to 0 for \( i = 1, \ldots, g-1 \) one obtains the following simpler version of Conjecture 9.1.

Conjecture 9.2. Let \( k, g \in \mathbb{Z}^+ \) be such that \( k \geq g \). Then, for \( p \geq 0 \),

\[
|\triangle F_{k+1}(gp)| + |\nabla_N^{(k)} F_{k-g}(p)| \geq (g + 1)p. \tag{9.1}
\]

Equality holds if and only if \( p = 0 \).

It was possible to verify by exhaustive computations that Conjecture 9.2 holds for values of \( n \leq 20 \) for \( g \leq 3 \). In these cases, computations showed that the inequality in (9.1) is tightest for \( g = 1 \) which is the situation in Theorem 7.1.(ii). However Conjecture 9.2 could not be proven. An attempt has been made to prove Conjecture 9.2 using similar techniques to the ones used to prove Theorem 7.1. The proof was not completed due to some technical details which although minor could not satisfactorily be dealt with. Also, the case discussion involved in the proof was very long and complex. The following conjecture could not be proven either. Exhaustive computations show that it holds for values of \( n \leq 20 \) for \( g = 2 \).

Conjecture 9.3. Let \( k, g \in \mathbb{Z}^+ \) be such that \( k \geq g \). Then, for \( p \geq 0 \),

\[
|\triangle^{(k)} F_{k+g}(p)| + |\nabla_N^{(k)} F_{k-g}(p)| \geq 2p.
\]

Equality holds if and only if \( p = 0 \).
Conjecture 9.3 is yet another possible way of generalising Theorem 7.1. It is thought that proving the truth of Conjecture 9.1 would help in proving the truth of the flat antichain conjecture. We have however been unable to prove that the flat antichain conjecture holds under the assumption that Conjecture 9.1 holds. While attempting to do so we tried to answer the following open problem.

**Open Problem 9.4.** Let $A$ be a squashed antichain on $[n]$ with parameters $p_i$. Let $k \in \mathbb{Z}^+$ and let $l$ be the smallest integer for which $p_l \neq 0$. Assume that $k > l$. The problem is to determine if the inequality

$$\left| \nabla_N^{(k)} A \right|$$

$$\geq \left| \nabla_N F_k - 1 \left[ p_{k-1} + \nabla_N F_{k-2} (p_{k-2} + \nabla_N F_{k-3} (p_{k-3} + \ldots + \nabla_N F_l (p_l)) ) \right] \right|$$

(9.2)

holds.

Open Problem 9.4 is easily solved for $l = k - 1$ as in this case $A^{(k-1)}$ is a collection $C_{k-1} (p_{k-1})$. Thus $\left| \nabla_N^{(k)} A \right| = \left| \nabla_N C_{k-1} (p_{k-1}) \right| \geq \left| \nabla_N F_k - 1 (p_{k-1}) \right|$ by Corollary 2.49. It is unknown if the same holds for values of $l$ smaller than $k - 1$. Note that the collection $\mathcal{F}^{(k-1)} A \subseteq \phi^{(k-1)} A$ of $(p_{k-1} + \left| \nabla_N^{(k)} A \right|) (k - 1)$-sets may not consist of consecutive $(k - 1)$-sets (see Example 1.9). So Corollary 2.49 cannot be applied to $\left| \nabla_N^{(k)} A \right| = \left| \nabla_N^{(k)} (A^{(k-1)} \cup \nabla_N^{(k-1)} A) \right|$. If one assumes that (9.2) and Conjecture 9.1 hold then the following theorem can be stated.

**Theorem 9.5.** Assume that both Conjecture 9.1 and (9.2) hold. Let $A$ be a squashed and non-flat antichain with $\overline{A}$ an integer and $|A| \leq \overline{A} + 1$ for each $A \in \mathcal{A}$. Then

$$\left| \phi^{|A|} A \right| > |A|.$$

*Proof.* Assume that both Conjecture 9.1 and (9.2) hold. Let $A$ be as in the statement of the theorem and let the integers $p_i$ be the parameters of $A$ with $l$ the smallest integer for which $p_l \neq 0$. Let $k = \overline{A}$, then $p_{k+1} = \sum_{i=1}^{k-l} i p_{k-i}$. This implies that
Chapter 9. The 3-Levels Result and the Flat Antichain Conjecture: Discussion

\[ |A| = \sum_{i=0}^{k-1} (i+1)p_{k-i}. \]

Now,

\[ |\phi^{(k)} A| = |\Delta F_{k+1}(p_{k+1})| + |A^{(k)}| + |\nabla_N^{(k)} A| \]

(as \( A \) is an antichain)

\[ = |\Delta F_{k+1}(p_{k-1} + 2p_{k-2} + 3p_{k-3} + \ldots + (k-l)p_l)| + p_k + |\nabla_N^{(k)} A| \]

\[ \geq |\Delta F_{k+1}(p_{k-1} + 2p_{k-2} + 3p_{k-3} + \ldots + (k-l)p_l)| + p_k \]

\[ + |\nabla_N F_{k-1}[p_{k-1} + |\nabla N F_{k-2}(p_{k-2} + |\nabla N F_{k-3}(p_{k-3} + \ldots \]

\[ + |\nabla N F_l(p_l)|]|]| \]

(by (9.2))

\[ > 2p_{k-1} + 3p_{k-2} + 4p_{k-3} + \ldots + (k-l+1)p_l \]

(by Conjecture 9.1 with \( g = k-l \))

\[ = |A|. \]

\[ \square \]

Theorem 9.5 is a counterpart of Corollary 8.12 which involves antichains \( A \) with \( \overline{A} \) an integer and having no sets on levels below \( \overline{A} - 1 \). Corollary 8.12 together with Theorem 9.5 leads to Conjecture 9.7 which is presented in the next section. There the \( \overline{A} \)-image of a squashed antichain \( A \) with \( \overline{A} \) an integer is discussed in more detail.

We conclude this section with another open problem which is partially related to the inequalities in Theorem 7.1.

Open Problem 9.6. For \( k, p \in \mathbb{Z}^+ \) and \( k \) fixed, determine the greatest lower bound for the function

\[ f(p) = |\Delta F_k(p)| + |\nabla_N F_k(p)|. \]

That \( |\Delta F_k(p)| + |\nabla_N F_k(p)| \geq p \) for all \( p \) is shown by Lemma 2.63 since \( |\Delta_N B| + |\nabla_N B| \geq 1 \) for any set \( B \). However \( p \) is not a tight lower bound. For any set \( B \), there is an implicit relationship between \( |\Delta_N B| \) and \( |\nabla_N B| \) as shown by Lemmas 2.60 and 2.62. Also, for any \( p \), \( |\Delta F_k(p)| \) and \( |\nabla_N F_k(p)| \) can be computed from the \( k \)-binomial representation of \( p \) by Theorems 2.39 and 2.56. These two observations may help in answering Open Problem 9.6.
9.3 \( \overline{\mathcal{A}} \)-Image of an Antichain \( \mathcal{A} \)

In this section is presented and discussed a conjecture in relation to the size of the \( \overline{\mathcal{A}} \)-image of a squashed antichain \( \mathcal{A} \) with \( \overline{\mathcal{A}} \) an integer.

**Conjecture 9.7.** Let \( \mathcal{A} \) be a squashed and non-flat antichain with \( \overline{\mathcal{A}} \) an integer. Then

\[
| \phi(\overline{\mathcal{A}}) \mathcal{A} | > | \mathcal{A} |. \tag{9.3}
\]

Corollary 8.12 is a special case which shows that Conjecture 9.7 holds for squashed antichains \( \mathcal{A}, \overline{\mathcal{A}} \in \mathbb{Z}^+ \), having no sets on levels below \( \overline{\mathcal{A}} - 1 \). Theorem 9.5 shows that if (9.2) and Conjecture 9.1 can be proved to hold, then Conjecture 9.7 holds for squashed antichains \( \mathcal{A}, \overline{\mathcal{A}} \in \mathbb{Z}^+ \), having no sets on levels above \( \overline{\mathcal{A}} + 1 \). Since Corollary 8.12 and Theorem 9.5 are corollaries of Theorem 8.10 and Conjecture 9.1 respectively, and Theorem 8.10 and Conjecture 9.1 are two possible generalisations of Theorem 7.1, Conjecture 9.7 can also be seen as a further generalisation of Theorem 7.1.

As for Conjecture 9.1, it is believed that the truth of Conjecture 9.7 may help in proving the flat antichain conjecture, even though we have been unable to make any progress in that direction.

Note that \( |\mathcal{A}| = |\phi(\overline{\mathcal{A}}) \mathcal{A}^*| \) where \( \mathcal{A}^* \) is any collection of \( |\mathcal{A}| \overline{\mathcal{A}} \)-sets. So Conjecture 9.7 states that the size of the \( \overline{\mathcal{A}} \)-image of a squashed antichain \( \mathcal{A}, \overline{\mathcal{A}} \in \mathbb{Z}^+ \), is uniquely minimised when \( \mathcal{A} \) consists of \( |\mathcal{A}| \overline{\mathcal{A}} \)-sets. We note that the condition that \( \overline{\mathcal{A}} \) is an integer is a necessary condition. That is, Conjecture 9.7 cannot be stated in terms of the \( |\mathcal{A}| \)-images of \( \mathcal{A} \) and \( \mathcal{A}^* \) with \( |\mathcal{A}| \neq \overline{\mathcal{A}} \) where \( \mathcal{A}^* \) is the squashed flat counterpart of \( \mathcal{A} \). This is shown in the next example.

**Example 9.8.** Let \( \mathcal{A} \) and \( \mathcal{A}^* \) be squashed antichains in \( \Lambda_{7,17} \) with \( \mathcal{A} = \{1234, 1235, 1245, 1345, 2345, 126, 136, \ldots, 456, 17, 27\} \) and \( \mathcal{A}^* = \{1234, 1235, 1245, 345, 126, 136, \ldots, 456, 127, 137, 237\} \). Note that \( \mathcal{A}^* \) is a flat counterpart of \( \mathcal{A} \) and that \( |\mathcal{A}| \neq \overline{\mathcal{A}} \).

Then \( \phi(\overline{\mathcal{A}}) \mathcal{A} \) consists of the \( \binom{6}{3} \) 3-subsets of \( [6] \) and the set 127, and \( \phi(3) \mathcal{A}^* = \phi(3) \mathcal{A} \cup \{137, 237\} \).
Conjecture 9.1 can be related to Theorems 4.2 and 5.1. The relationship with Theorem 5.1 is explored in the next section. It is easy to see that the truth of Conjecture 9.7 implies Theorem 4.2. Let \( A \) be a non-flat antichain on \([n]\) with \( \mathcal{A} \in \mathbb{Z}^+ \). \( A \) can be assumed to be squashed by Theorem 2.78. If Conjecture 9.7 is true, then the size of the \( \mathcal{A} \)-image of \( A \) is strictly larger than \( |A| \). By Lemma 2.80, it follows that there exists a flat antichain consisting of \( |A| \mathcal{A} \)-subsets of \([n]\). This shows that Theorem 4.2 holds.

### 9.4 Conjecture 9.7 and Theorem 5.1

In this section we explore a relationship between Conjecture 9.7 and Theorem 5.1. Conjecture 9.7 can be viewed as a generalisation of Theorem 5.1. Consider the collection \( F_k \left( \binom{n}{k} \right) \) of all \( k \)-subsets of \([n]\) in squashed order. Let \( p \in \mathbb{Z}^+ \) with \( p < \binom{n}{k} \) and let \( A \) be a squashed and non-flat antichain \( A \) of size \( p \) and volume \( pk \). Assume that the problem is to replace \( F_k(p) \) by \( A \) such that \( A \cup \left( F_k \left( \binom{n}{k} \right) \setminus F_k(p) \right) \) is an antichain. Note that \( F_k(p) \) is the squashed flat counterpart of \( A \) and that what we propose to do here is to perform the reverse operation to that of replacing an antichain by one of its flat counterparts. Using Theorem 5.1 and assuming that Conjecture 9.7 holds, we show that such an operation is not possible if, in addition, it is required that \( A \cup \left( F_k \left( \binom{n}{k} \right) \setminus F_k(p) \right) \) is an antichain.

Assume that \( p = \binom{m}{k} \) for some \( m < n \). Then \( A \) is a non-flat antichain of size \( \binom{m}{k} \) with \( \mathcal{A} = k \). By Theorem 5.1 the unique antichain on \([m]\) of size \( \binom{m}{k} \) and with average set size size \( k \) is the collection \([m]^k\). Since \( A \) is not flat it follows that \( A \) is not an antichain on \([m]\) but is an antichain on \([m']\) with \( m' > m \). Therefore \( \phi^{(k)} A \cap \left( F_k \left( \binom{m'}{k} \right) \setminus F_k \left( \binom{m}{k} \right) \right) \neq \emptyset \). As \( A \) is squashed, it follows that there are sets in \( F_k \left( \binom{m'}{k} \right) \setminus F_k \left( \binom{m}{k} \right) \) which are in the shade of some sets in \( A \). We conclude that \( A \cup \left( F_k \left( \binom{m}{k} \right) \setminus F_k(p) \right) \) is not an antichain.

Assume that \( p \neq \binom{m}{k} \) for any \( m < n \) and that Conjecture 9.7 holds. Note that \( o^{\mathcal{A}} A = o^{(k)} A \). By Conjecture 9.7, \( o^{(k)} A \cap \left( F_k \left( \binom{n}{k} \right) \setminus F_k(p) \right) \neq \emptyset \). As \( A \) is squashed, it follows that there are sets in \( F_k \left( \binom{n}{k} \right) \setminus F_k(p) \) which are in the new-shade of some
sets in \(A\). We conclude that \(A \cup (F_k(\binom{n}{k}) \setminus F_k(p))\) is not an antichain.

In summary, we have shown, using Theorem 5.1, that for \(p = \binom{n}{k}\) for some \(m < n\), the collection \(A \cup (F_k(\binom{n}{k}) \setminus F_k(p))\) is not an antichain. Assuming that Conjecture 9.7 holds, this is also the case for any \(p < \binom{n}{k}\). It is in this sense that Conjecture 9.7 can be seen as a generalisation of Theorem 5.1.

### 9.5 \([\overline{A}]\)-Projection of an Antichain \(A\)

We now turn to the problem of minimising the size of the \([\overline{A}]\)-projection of an antichain. We state the last conjecture in this chapter.

**Conjecture 9.9.** Let \(A\) be a non-flat antichain on \([n]\) and let \(A^*\) be the squashed flat counterpart of \(A\). Then

\[
|\hat{\psi}(\overline{A})A| > |\hat{\psi}(\overline{A})A^*|.
\]

Several results in Chapter 8 show that Conjecture 9.9 holds in certain cases.

Let \(A\) be a non-flat antichain on \([n]\) with parameters \(p_i\). If \(|A| \geq |\overline{A}|\) for each \(A \in \mathcal{A}\), then Theorem 8.3 says that Conjecture 9.9 holds for \(A\). Let \(h\) be the largest integer for which \(p_i \neq 0\) and let \(k = |\overline{A}|\). If \(\sum_{i=1}^{h-k} i p_{k+i} \leq \binom{n}{k+1}\) then Theorem 8.15 shows that Conjecture 9.9 holds for \(A\). Let \(l\) be the smallest integer for which \(p_i \neq 0\). If \(h = k + 2\) and \(l = k - 1\) then Corollary 8.21 shows that Conjecture 9.9 holds for \(A\). It can be seen that there is a strong suspicion that the size of the \([\overline{A}]\)-projection of \(A\) is uniquely minimised when \(A\) is squashed and flat. For completeness, we must add that it is not too difficult to show (the proof is not given here) that \(|\hat{\psi}(\overline{A})A| \geq |\hat{\psi}(\overline{A})A^*|\) when \(A\) is the antichain in Corollary 8.4 with \(|\overline{A}| \neq \overline{A}\). That is, when \(|A| \leq \overline{A} = \overline{A} + 1\) for each \(A \in \mathcal{A}\). However, it has not been possible so far to prove that the strict inequality holds in this case.

It may be the case that the sizes of all \(l\)-projections of \(A\), \(1 \leq l \leq n\), are minimised when \(A\) is squashed and flat. We have not been able to prove this, even when assuming that Conjecture 9.9 holds. We state this as an open problem.
Open Problem 9.10. Let \( \mathcal{A} \) be a non-flat antichain on \([n]\) and let \( \mathcal{A}^* \) be the squashed flat counterpart of \( \mathcal{A} \). Determine whether the inequality
\[
|\bigtriangleup^l(\mathcal{A})| \geq |\bigtriangleup^l(\mathcal{A}^*)|
\]
holds for each \( l, \ 0 \leq l \leq n \).

Note that Conjecture 9.9 is a sufficient but not a necessary condition for the flat antichain conjecture to hold. Assume, for some antichains \( \mathcal{A} \) and \( \mathcal{A}^* \) defined as in Conjecture 9.9, that \( |\bigtriangleup^{(k)}(\mathcal{A})| \leq |\bigtriangleup^{(k)}(\mathcal{A}^*)| \). More precisely, assume that \( |\bigtriangleup^{(k)}(\mathcal{A})| < |\bigtriangleup^{(k)}(\mathcal{A}^*)| \). If \( n \) is large enough for \( \mathcal{A}^* \) to be an antichain on \([n]\), then the flat antichain conjecture holds for \( \mathcal{A} \). We conclude the section with the following lemma.

Lemma 9.11. Assume that Conjecture 9.9 holds. Then Conjecture 5.18 holds.

**Proof.** Assume that Conjecture 9.9 holds. Let \( \mathcal{A} \in \Lambda_{n,3} \) be flat and full with sets on levels \( k \) and \( k + 1 \). Let \( \mathcal{B} \in \Lambda_{n,3} \) be any non-flat antichain with \( V(\mathcal{A}) = V(\mathcal{B}) \). Then \( |\bigtriangleup^{(k)}(\mathcal{B})| > |\bigtriangleup^{(k)}(\mathcal{A})| \) by Conjecture 9.9 since \( |\mathcal{A}| = k \). As \( \mathcal{A} \) is full and consists of \( (k + 1) \) and \( k \)-sets, \( |\bigtriangleup^{(k)}(\mathcal{A})| = \binom{n}{k} \). It follows that \( |\bigtriangleup^{(k)}(\mathcal{B})| > \binom{n}{k} \) and \( \mathcal{B} \) is not an antichain on \([n]\). Hence \( \mathcal{A} \) is a profile-unique antichain in \( \Lambda_{n,3} \) with volume \( V(\mathcal{A}) \) and Conjecture 5.18 holds. \( \square \)

9.6 Other Attempts at Solving the Flat Antichain Conjecture

In this section two other approaches taken to solve the flat antichain conjecture are presented. The first approach consists of determining a necessary and sufficient condition for any antichain on \([n] \) to have a flat counterpart which is an antichain on \([n] \).

Theorem 7.1 is not a necessary condition for the flat antichain conjecture to hold as was made clear in the alternative proof of Corollary 8.5. Let \( \mathcal{A} \) be a non-flat antichain on \([n] \) on three levels, \( k + 1, k, \) and \( k - 1 \). Let the integers \( p_i \) be the parameters of \( \mathcal{A} \)
and assume that $p_{k+1} > p_{k-1}$. Let $\mathcal{B} = L(p_{k-1}, \mathcal{A}^{(k+1)})$. A necessary and sufficient condition for obtaining a flat counterpart of $\mathcal{A}$ which is an antichain on $[n]$ is that $|\triangle_N \mathcal{B}| + |\nabla \mathcal{A}^{(k-1)}| \geq 2p_{k-1}$ so that the sets in $\mathcal{B}$ and $\mathcal{A}^{(k-1)}$ can be replaced by $2p_{k-1}$ $k$-sets in $|\triangle_N \mathcal{B}| + |\nabla \mathcal{A}^{(k-1)}|$ while retaining the antichain property. As seen in the alternative proof of Corollary 8.5, Theorem 7.1 is a sufficient condition for this set replacement to be possible, since $|\triangle_N \mathcal{B}| + |\nabla \mathcal{A}^{(k-1)}| \geq |\triangle_N L_{k+1}(p_{k-1})| + |\nabla L_{k-1}(p_{k-1})| > 2p_{k-1}$ by Corollary 2.37 and Theorems 2.47 and 7.1.(i).

It is thus of interest to prove that $|\triangle_N \mathcal{B}| + |\nabla \mathcal{A}^{(k-1)}| \geq 2p_{k-1}$ in all cases without having to rely on applying Theorem 7.1. This was the approach taken, in a more general setting, by Roberts [27] to prove Theorem 4.3.

The second approach taken to solve the flat antichain conjecture consists of investigating the volumes which a flat antichain in $\Lambda_n,s$ can achieve. To do this, one attempts to build squashed and flat collections $\mathcal{C}_i$ of size $s$ and volumes $V_{\min}(\Lambda_{n,s})$, $V_{\min}(\Lambda_{n,s}) + 1$, $\ldots$, $V_{\min}(\Lambda_{n,s}) + i$, $\ldots$, $V_{\max}(\Lambda_{n,s})$ respectively. If $\mathcal{C}_i$ is not an antichain on $[n]$ for some $i$, then one must show that there exists no $\mathcal{A} \in \Lambda_{n,s}$ with $V(\mathcal{A}) = V_{\min}(\Lambda_{n,s}) + i$. For if such an $\mathcal{A}$ exists then $\mathcal{A}$ is necessarily non-flat, thus disproving the flat antichain conjecture. So far we have been unsuccessful at proving that such an antichain $\mathcal{A}$ does not exist. Conjecture 5.18 arose from these attempts. If $\mathcal{C}_i$ is not an antichain, then there is a subcollection $\mathcal{D}_i \subset \mathcal{C}_i$ which is a flat and full antichain on $[n]$ (but not of size $s$). If Conjecture 5.18 holds then $\mathcal{D}_i$ is a unique-profile and flat antichain in $\Lambda_{n,s}$ with volume $V(\mathcal{D}_i) < V(\mathcal{C}_i) = V_{\min}(\Lambda_{n,s}) + i$. It was hoped that this fact could shed some light on whether or not an antichain with volume $V_{\min}(\Lambda_{n,s}) + i$ could exist in $\Lambda_{n,s}$. This approach proved unsuccessful.

### 9.7 Conclusion

While Theorem 7.1 is a strong result which was used to show that the flat antichain conjecture holds in six special cases, it seems that more is required to prove the conjecture in its entirety, at least if pursuing the approach taken in Chapter 8. The condition for the flat antichain conjecture to hold as expressed in Theorem 8.14 might
be worth considering. We repeat the condition here.

Let $\mathcal{A}$ be a non-flat antichain on $[n]$ with $|\mathcal{A}| = k$ and with parameters $p_i$. Let $h$ be the largest integer for which $p_i \neq 0$. The condition for the flat antichain conjecture to hold for $\mathcal{A}$ is that $\sum_{i=1}^{h-k} ip_{k+i} \leq \binom{n}{k+1}$. The proof of Theorem 8.19, which involves antichains on four consecutive levels, shows how the derivation can become intricate when the condition $\sum_{i=1}^{h-k} ip_{k+i} \leq \binom{n}{k+1}$ is not met. It has been impossible so far to generalise the arguments used in the proof of Theorem 8.19, and it seems likely that such a generalisation cannot be made.

Another approach to solve the flat antichain conjecture would be to ask, for an antichain $\mathcal{A}$ with $|\mathcal{A}| = k$, if it is possible that the condition $\sum_{i=1}^{h-k} ip_{k+i} \leq \binom{n}{k+1}$ is not met. We thus conclude the chapter by the following open problem.

**Open Problem 9.12.** Let $\mathcal{A}$ be a non-flat antichain on $[n]$ with $|\mathcal{A}| = k$ and with parameters $p_i$. Let $h$ be the largest integer for which $p_i \neq 0$. The problem is to determine if the inequality

$$\sum_{i=1}^{h-k} ip_{k+i} \leq \binom{n}{k+1}$$

holds.

The answer to this problem is unknown in general.
Chapter 10

The Flat Antichain Conjecture: Some Consequences
10.1 Introduction

This chapter presents new results based upon the assumption that the flat antichain conjecture holds for all antichains on \([n]\). Theorem 10.1 states a necessary and sufficient condition for the existence of antichains with a given size and volume. An application to the existence of \((n, k)\)CSSs is also shown in this section. Section 10.3 discusses volumes of antichains and defines a new equivalence relation in \(\Lambda_{n,s}\). Recall that for given \(n\) and \(s\), (i) and (ii) of Theorem 5.4 determine the minimum volume and minimum size ideal an antichain in \(\Lambda_{n,s}\) can achieve. In Section 10.4 it is shown how the assumption that the flat antichain conjecture holds provides an alternative proof of Theorem 5.4.(i). Further, a partial answer to Open Problem 5.17 is given. Open Problem 5.17 is concerned with a possible relationship between Theorem 5.4.(i) and Theorem 5.4.(ii).

10.2 Existence of Antichains with Given Size and Volume

If the flat antichain conjecture holds, then testing if the integers \(s\) and \(V\) are the size and volume of an antichain on \([n]\) is equivalent to testing if the parameters \(q_i\) of a flat collection of sets with size \(s\) and volume \(V\) are the parameters of an antichain on \([n]\). This is formalised in Theorem 10.1. Note that it has already been shown that the \(q_i\)s verify the LYM inequality (see Theorem 4.9).

**Theorem 10.1.** Assume that the flat antichain conjecture holds. Let \(n\), \(s\) and \(V \in \mathbb{Z}^+\) and let \(k\) and \(r\) be the unique non negative integers such that \(V = ks + r\) and \(0 \leq r < s\). Then \(s\) and \(V\) are the size and volume of an antichain on \([n]\) if and only if

\[
|\Delta F_{k+1}(r)| + s - r \leq \binom{n}{k}.
\]  

(10.1)

**Proof.** Assume that the flat antichain conjecture holds and let \(s\), \(V\), \(k\) and \(r\) be as in the statement of the theorem. Then the parameters \(q_i\) of any flat collection of sets of
size $s$ and volume $V$ are $q_{k+1} = r$, $q_k = s - r$, $q_i = 0$ for $i \neq k+1, k$. By the truth of the flat antichain conjecture there exists an antichain on $[n]$ with size $s$ and volume $V$ if and only if the $q_i$s are the parameters of an antichain on $[n]$. It follows that by Theorem 2.77 there exists an antichain on $[n]$ with size $s$ and volume $V$ if and only (10.1) holds.

Let $\mathcal{C}$ be a $(n, k)CSS$ and let $R \in \mathbb{Z}^+$. We have seen in Section 4.2 that a necessary condition for $|\mathcal{C}| = R$ is that there exists an antichain on $[R]$ of size $n$ and volume $kR$. Assuming that the flat antichain conjecture holds, Theorem 10.1 can be used to find a necessary condition for the existence of a $(n, k)CSS$ with a given size.

**Corollary 10.2.** Assume that the flat antichain conjecture holds. Let $n$, $k$, $R \in \mathbb{Z}^+$ and let $l$ and $r$ be the unique non negative integers such that $kR = ln + r$ and $0 \leq r < n$. Then $R$ is the size of a $(n, k)CSS$ only if

$$|\Delta F_{i+1}(r)| + n - r \leq \binom{R}{l}.$$  

**Proof.** Assume that the flat antichain conjecture holds. Let $n$, $k$ and $R$ be as in the statement of the corollary and assume that a $(n, k)CSS$ of size $R$ exists. By Observations 2.89 and 2.87, it follows that there exists an antichain on $[R]$ of size $n$ and volume $kR$. The result follows from Theorem 10.1.

### 10.3 Volumes of Antichains

If for all $n$, $s \in \mathbb{Z}^+$ and each integer $V$, $V_{min}(\Lambda_{n,s}) \leq V \leq V_{max}(\Lambda_{n,s})$, $V$ is a volume of a flat antichain in $\Lambda_{n,s}$, then the truth of the flat antichain conjecture would immediately follow. But this is not true in general as the following example demonstrates.

**Example 10.3.** In Example 4.8 we have seen that there exists no flat antichain in $\Lambda_{5,5}$ with volume 18.

Consider $\Lambda_{8,8}$. There exists no flat antichain in $\Lambda_{8,8}$ with volume 9 or 10.
Chapter 10. The Flat Antichain Conjecture: Some Consequences

Let $V$ be an integer which is not the volume of a flat antichain in $\Lambda_{n,s}$. If there exists $\mathcal{A} \in \Lambda_{n,s}$ with $V(\mathcal{A}) = V$ then the flat antichain conjecture does not hold. Alternatively, we have

**Corollary 10.4.** Assume that the flat antichain conjecture holds. Let $V \in \mathbb{Z}^+$. If there does not exist a flat antichain $\mathcal{A}^* \in \Lambda_{n,s}$ with $V(\mathcal{A}^*) = V$, then, for each $\mathcal{A} \in \Lambda_{n,s}$, $V(\mathcal{A}) \neq V$.

Under the assumption of the truth of the flat antichain conjecture, Example 10.3 and Corollary 10.4 show that there are volumes which cannot be achieved by an antichain in $\Lambda_{n,s}$. This is to be contrasted with Theorem 6.5. By Theorem 6.5 there exists a flat antichain on $[n]$ with volume $V$ for each $V < U_n$, where $U_n$ is as defined in Definition 6.4. In Theorem 6.5 however, no mention of the size of the flat antichain is made.

**Example 10.5.** Referring to Example 10.3, there is no flat antichain in $\Lambda_{5,8}$ with volume 18 and there is no flat antichain in $\Lambda_{8,8}$ with volume 9 or 10. For $n = 5$, $U_5 = 22$, and for $n = 8$, $U_8 = 251$ (see Example 6.26). By Theorem 6.5 there exists a flat antichain $\mathcal{A}$ on $[5]$ with $V(\mathcal{A}) = V$ for each $V < 22$ and there exists a flat antichain $\mathcal{B}$ on $[8]$ with $V(\mathcal{B}) = V$ for each $V < 251$. Note that $\mathcal{A}$ and $\mathcal{B}$ do not need to belong to $\Lambda_{5,8}$ and $\Lambda_{8,8}$ respectively.

If the flat antichain conjecture holds, then it is natural to define the following equivalence relation.

**Definition 10.6.**

Let $\mathcal{A}, \mathcal{B} \in \Lambda_{n,s}$. We say that $\mathcal{A}$ is **volume-equivalent** to $\mathcal{B}$, written $\mathcal{A} \equiv_{V(\Lambda_{n,s})} \mathcal{B}$, if $V(\mathcal{A}) = V(\mathcal{B})$. The equivalence class of $\mathcal{A} \in \Lambda_{n,s}$ is denoted by $[\mathcal{A}]_V$, where $V = V(\mathcal{A})$.

An easy consequence of the flat antichain conjecture is the following.

**Corollary 10.7.** Assume that the flat antichain conjecture holds. Then, for each $\mathcal{A} \in \Lambda_{n,s}$, there exists $\mathcal{A}^* \in [\mathcal{A}]_V$ such that $\mathcal{A}^*$ is flat.
### 10.4 Minimum Size Ideal and Minimum Volume

Theorems 5.4.(i) and (ii) respectively give the values of the minimum volume and the minimum size ideal which can be achieved by an antichain in \( \Lambda_{n,s} \). Assuming that the flat antichain conjecture holds it is possible to give an alternative proof of Theorem 5.4.(i).

**Alternative Proof of Theorem 5.4.(i).** Assume that the flat antichain conjecture holds. Consider the class \([\mathcal{A}]_{V_{\min}(\Lambda_{n,s})}\) under the relation \(\equiv_{V(\Lambda_{n,s})}\) in \( \Lambda_{n,s} \). Certainly, \([\mathcal{A}]_{V_{\min}(\Lambda_{n,s})} \neq \emptyset\). Then there exists a flat antichain \( \mathcal{A}^* \in [\mathcal{A}]_{V_{\min}(\Lambda_{n,s})}\) by Corollary 10.7. As \( V(\mathcal{A}^*) = V_{\min}(\Lambda_{n,s}) \), \( \mathcal{A}^* \) must be non-reducible and full. Let \( k \in \mathbb{N} \) be such that \( \binom{n}{k} < s \leq \binom{n}{k+1} \). Then, by Lemma 5.11.(i), the parameters \( q_i \) of \( \mathcal{A}^* \) are such that \( q_i = 0 \) for \( i \neq k + 1, k \), and \( q_{k+1} \) is the smallest solution of (5.2). It follows that \( V_{\min}(\Lambda_{n,s}) = V(\mathcal{A}^*) = q_{k+1}(k + 1) + (s - q_{k+1})k \). This proves Theorem 5.4.(i).  

In Open Problem 5.17 Clements asked whether Theorem 5.4.(i) could be derived from Theorem 5.4.(ii) and vice versa. A partial answer to this problem can be given provided that the flat antichain conjecture holds.

**Corollary 10.8.** Assume that the flat antichain conjecture holds. Then Theorem 5.4.(ii) implies Theorem 5.4.(i).

**Proof.** Assume that the flat antichain conjecture holds. Let \( n, s \in \mathbb{Z}^+ \) be given, and let \( k \in \mathbb{N} \) be such that \( \binom{n}{k} < s \leq \binom{n}{k+1} \). Assume that Theorem 5.4.(ii) holds. Let \( \mathcal{A} \) be an antichain with parameters \( p_i \) which are such that \( p_i = 0 \) for \( i \neq k + 1, k \), and \( p_{k+1} \) is the smallest solution of (5.2). Then \( |Z\mathcal{A}| = I_{\min}(\Lambda_{n,s}) \) by Theorem 5.4.(ii).

We show that \( V(\mathcal{A}) = V_{\min}(\Lambda_{n,s}) \). Assume that this is not the case and let \( \mathcal{A}_1 \in \Lambda_{n,s} \) with \( V(\mathcal{A}_1) = V_{\min}(\Lambda_{n,s}) < V(\mathcal{A}) \). Let \( \mathcal{A}^* \) be a flat counterpart of \( \mathcal{A}_1 \). Then \( \mathcal{A}^* \in \Lambda_{n,s} \) by the truth of the flat antichain conjecture and \( \mathcal{A}^* \in [\mathcal{A}]_{V_{\min}(\Lambda_{n,s})} \). This implies that \( \mathcal{A}^* \) must be non-reducible and full. By Lemma 5.11.(i), the parameters \( q_i \) of \( \mathcal{A}^* \) are such that \( q_i = 0 \) for \( i \neq k + 1, k \), and \( q_{k+1} \) is the smallest solution of (5.2).
It follows that $\mathcal{A}^* \cong \mathcal{A}$ and $V(\mathcal{A}^*) = V(\mathcal{A}_1) = V(\mathcal{A})$, contrary to the assumption. Therefore $V(\mathcal{A}) = V_{\min}(\Lambda_{n,s})$ and Theorem 5.4(i) holds. \hfill \Box

The converse of Corollary 10.8 has not been proved to date. One could be tempted to prove that Theorem 5.4(ii) implies Theorem 5.4(i) by using a similar argument to the one used in the proof of Corollary 10.8. However, such an approach will not work as it would require that the following assertion holds.

Let $\mathcal{A} \in \Lambda_{n,s}$ and let $\mathcal{A}^*$ be a flat counterpart of $\mathcal{A}$. Assume that $\mathcal{A}^* \in \Lambda_{n,s}$. Then $|I_A| \geq |I_{\mathcal{A}^*}|$.

This assertion is not true in general as the following example demonstrates.

**Example 10.9.** We take the antichains in Example 9.8. Let $\mathcal{A}$ and $\mathcal{A}^*$ be squashed antichains in $\Lambda_{7,17}$ with $\mathcal{A} = \{1234, 1235, 1245, 1345, 2345, 126, 136, \ldots, 456, 17, 27\}$ and $\mathcal{A}^* = \{1234, 1235, 1245, 345, 126, 136, \ldots, 456, 127, 137, 237\}$. Note that $\mathcal{A}^*$ is a flat counterpart of $\mathcal{A}$.

$I_A$ contains the $\binom{4}{4}$ 4-subsets of [5], the $\binom{3}{3}$ 3-subsets of [6], the $\binom{2}{2}$ 2-subsets of [6], the sets 17 and 27, the $\binom{1}{1}$ singletons of [7], and $\emptyset$. Hence $|I_A| = \binom{4}{4} + \binom{3}{3} + \binom{2}{2} + 2 + 7 + 1 = 50$. $I_{\mathcal{A}^*}$ contains 3 4-sets, the $\binom{3}{3}$ 3-subsets of [6], the sets 127, 137 and 237, the $\binom{2}{2}$ 2-subsets of [6], the sets 17, 27, and 37, the $\binom{1}{1}$ singletons of [7], and $\emptyset$. Then $|I_{\mathcal{A}^*}| = 3 + \binom{3}{3} + 3 + \binom{2}{2} + 3 + 7 + 1 = 52 > |I_A|$.
Chapter 11

Conclusion
Chapter 11. Conclusion

The two main lines of investigation pursued in this thesis are the study of the volumes of antichains and the study of the sizes of the new-shadows and new-shades of appropriately chosen collections of sets. Some progress has been made towards solving the flat antichain conjecture as shown in Theorems 6.7, 8.2, 8.7, 8.14, 8.19, and Corollaries 8.4 and 8.5.

We have seen in Chapter 5 that the antichains in \( \Lambda_{n,s} \) which achieve minimum (maximum) volume \( V_{\min}(\Lambda_{n,s}) \) (\( V_{\max}(\Lambda_{n,s}) \)) in \( \Lambda_{n,s} \) are profile-unique and flat. That is, \( V_{\min}(\Lambda_{n,s}) \) and \( V_{\max}(\Lambda_{n,s}) \) are volumes which in \( \Lambda_{n,s} \) are only achieved by a flat antichain. This suggests that some other volumes of antichains in \( \Lambda_{n,s} \) may only be achieved by antichains in \( \Lambda_{n,s} \) which are flat. This is stated in Conjecture 5.18. In Chapter 6 it is shown that given any antichain \( A \) on \([n]\) there exists a flat antichain on \([n]\) with volume \( V(A) \). In doing so the flat antichain conjecture is proven to hold in one case (see Theorem 6.7).

The 3-levels result (Theorem 7.1) in Chapter 7 determines a new relationship for the sizes of the new-shadows and new-shades of appropriately chosen collections of sets. Theorem 7.1 is very powerful and enables one to solve the flat antichain conjecture in six cases, namely Theorems 8.2, 8.7, 8.14, 8.19 and Corollaries 8.4 and 8.5. Theorem 7.1 could be generalised in one direction as shown by Theorem 8.10. Several conjectures arose while proving Theorem 7.1. We only mention Conjectures 9.7 and 9.9 here. Conjecture 9.7 is another generalisation of Theorem 7.1: It says that the \( \overline{A} \)-image of a squashed antichain \( A \) with \( \overline{A} \) an integer is uniquely minimised when \( A \) is flat. Conjecture 9.9 says that the \( [\overline{A}] \)-projection of an antichain \( A \) is uniquely minimised when \( A \) is squashed and flat. In Chapter 8 are presented some cases for which Conjectures 9.7 and 9.9 are shown to hold.

Also worth mentioning is Open Problem 9.12. If the inequality in Open Problem 9.12 holds for all antichains, then the flat antichain conjecture is true. We believe that a further investigation of Open Problem 9.12 and possibly Conjecture 9.7 might prove helpful in solving the flat antichain conjecture. Assuming that the flat antichain conjecture holds, necessary and sufficient conditions for the existence of antichains with given size and volume can be stated. This is shown in Theorem 10.1.
Appendix A

Table of Symbols
A, B, L denote subsets of \([n]\). B denotes a collection of subsets of \([n]\).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Z)</td>
<td>The set of integers.</td>
</tr>
<tr>
<td>(Z^+)</td>
<td>The set of positive integers.</td>
</tr>
<tr>
<td>(N)</td>
<td>(Z^+ \cup {0}).</td>
</tr>
<tr>
<td>(R)</td>
<td>The set of reals.</td>
</tr>
<tr>
<td>(R^+)</td>
<td>The set of positive reals.</td>
</tr>
<tr>
<td>([a, b])</td>
<td>The set of reals (x) such that (a \leq x \leq b).</td>
</tr>
<tr>
<td>([n])</td>
<td>The set ({1, \ldots, n}), (n \in Z^+).</td>
</tr>
<tr>
<td>([n]^k)</td>
<td>The collection of all the (k)-subsets of ([n]).</td>
</tr>
<tr>
<td>([x])</td>
<td>The greatest integer (\leq x).</td>
</tr>
<tr>
<td>([x])</td>
<td>The smallest integer (\geq x).</td>
</tr>
<tr>
<td>(E_1 \equiv E_2)</td>
<td>Expression (E_1) is equivalent to expression (E_2).</td>
</tr>
<tr>
<td>(n!)</td>
<td>(n \in N). (n! = n \times (n-1) \times \ldots \times 1) with (0! = 1).</td>
</tr>
<tr>
<td>(\binom{n}{k})</td>
<td>(n, k \in N). (\binom{n}{k} = \frac{n!}{(n-k)!k!}) for (n \geq k).</td>
</tr>
<tr>
<td>(\binom{x}{k})</td>
<td>(x \in R), (k \in N). (\binom{x}{k} = \binom{x-1}{k-1} + \binom{x-1}{k}) for (x \geq k).</td>
</tr>
<tr>
<td>(i \in B)</td>
<td>(i) is an element of (B).</td>
</tr>
<tr>
<td>(A \subseteq B)</td>
<td>(A) is a subset of (B).</td>
</tr>
<tr>
<td>(A \subset B)</td>
<td>(A) is a proper subset of (B).</td>
</tr>
<tr>
<td>(A \supseteq B)</td>
<td>(A) is a superset of (B).</td>
</tr>
<tr>
<td>(A \supset B)</td>
<td>(A) is a proper superset of (B).</td>
</tr>
<tr>
<td>(A \cap B)</td>
<td>The intersection of (A) and (B).</td>
</tr>
<tr>
<td>(A \cup B)</td>
<td>The union of (A) and (B).</td>
</tr>
<tr>
<td>(A \setminus B)</td>
<td>The set difference: ({i : i \in A, i \notin B}).</td>
</tr>
<tr>
<td>(A + B)</td>
<td>The symmetric difference: ((A \setminus B) \cup (B \setminus A)).</td>
</tr>
<tr>
<td>(B \uplus L)</td>
<td>({D : D = B \cup L, B \in B} ) where (B \cap L = \emptyset) for all (B \in B) and (b &lt; l) for all (b \in B, B \in B, l \in L).</td>
</tr>
<tr>
<td>([B])</td>
<td>The size or cardinality of (B).</td>
</tr>
<tr>
<td>(B^\prime)</td>
<td>The complement of (B): ([n] \setminus B).</td>
</tr>
<tr>
<td>([B])</td>
<td>The size or cardinality of (B).</td>
</tr>
<tr>
<td>(B^\prime)</td>
<td>The complement of (B): ({B : B^\prime \in B}). If the sets in (B = {B_1, B_2, \ldots, B_m}) are ordered, then (B^\prime = {B_1^\prime, B_2^\prime, \ldots, B_m^\prime}).</td>
</tr>
<tr>
<td>(B^{(i)})</td>
<td>({B : B \in B,</td>
</tr>
<tr>
<td>(V(B))</td>
<td>(\sum_{b \in B}</td>
</tr>
<tr>
<td>(\overline{B})</td>
<td>(\frac{V(B)}{</td>
</tr>
</tbody>
</table>

| Table A.1: Notation: 1 |
A, B denote subsets of $[n]$. $\mathcal{A}, \mathcal{B}$ denote collections of subsets of $[n]$.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{I}B$</td>
<td>${D : D \subseteq B, B \in \mathcal{B}}$</td>
</tr>
<tr>
<td>$\mathcal{I}^{(k)}B$</td>
<td>${D : D \in \mathcal{I}B,</td>
</tr>
<tr>
<td>$\mathcal{F}B$</td>
<td>${D \subseteq [n] : D \supseteq B, B \in \mathcal{B}}$</td>
</tr>
<tr>
<td>$\mathcal{F}^{(k)}B$</td>
<td>${D : D \in \mathcal{F}B,</td>
</tr>
<tr>
<td>$p_i$</td>
<td>The $i$th parameter of a collection of sets $\mathcal{B}$: $p_i =</td>
</tr>
<tr>
<td>$(p_1, \ldots, p_n)$</td>
<td>The profile of a collection of sets $\mathcal{B}$: The $(n+1)$-tuple of the parameters of $\mathcal{B}$.</td>
</tr>
<tr>
<td>$(n)$CSS</td>
<td>A completely separating system on $[n]$.</td>
</tr>
<tr>
<td>$R(n)$</td>
<td>The size of a minimal $(n)$CSS.</td>
</tr>
<tr>
<td>$(n,k)$CSS</td>
<td>A completely separating system $\mathcal{C}$ on $[n]$ with $</td>
</tr>
<tr>
<td>$R(n,k)$</td>
<td>The size of a minimal $(n,k)$CSS.</td>
</tr>
<tr>
<td>$\mathcal{A} \cong \mathcal{B}$</td>
<td>$\mathcal{A} \cong \mathcal{B}$ if the profile of $\mathcal{A}$ is equal to the profile of $\mathcal{B}$.</td>
</tr>
<tr>
<td>$S(k_1, k_2, \ldots, k_n)$</td>
<td>A multiset on $[n]$.</td>
</tr>
<tr>
<td>$\Lambda_n$</td>
<td>The collection of all the antichains on $[n]$.</td>
</tr>
<tr>
<td>$V_{\text{max}}(\Lambda_n)$</td>
<td>$\max_{\mathcal{A} \subseteq \Lambda_n} V(\mathcal{A})$.</td>
</tr>
<tr>
<td>$V_{\text{max}}(\Lambda_{n,s})$</td>
<td>$\max_{\mathcal{A} \subseteq \Lambda_{n,s}} {V : V = V(\mathcal{A}) \text{ where } \mathcal{A} = [n]^s, V &lt; V_{\text{max}}(\Lambda_n)}$.</td>
</tr>
<tr>
<td>$\Lambda_{n,r}$</td>
<td>The collection of all the antichains on $[n]$ of size $s$.</td>
</tr>
<tr>
<td>$V_{\text{min}}(\Lambda_{n,r})$</td>
<td>$\min_{\mathcal{A} \subseteq \Lambda_{n,r}} V(\mathcal{A})$.</td>
</tr>
<tr>
<td>$V_{\text{max}}(\Lambda_{n,r})$</td>
<td>$\max_{\mathcal{A} \subseteq \Lambda_{n,r}} V(\mathcal{A})$.</td>
</tr>
<tr>
<td>$I_{\text{min}}(\Lambda_{n,r})$</td>
<td>$\min_{\mathcal{A} \subseteq \Lambda_{n,r}}</td>
</tr>
<tr>
<td>$\mathcal{A} \equiv V(\Lambda_{n,r}) \mathcal{B}$</td>
<td>$\mathcal{A} \equiv V(\Lambda_{n,r}) \mathcal{B}$ if $\mathcal{A}, \mathcal{B} \in \Lambda_{n,r}$, and $V(\mathcal{A}) = V(\mathcal{B})$.</td>
</tr>
<tr>
<td>$[\mathcal{A}]_V$</td>
<td>The equivalence class of $\mathcal{A}$ under $\equiv V(\Lambda_{n,r})$ where $V(\mathcal{A}) = V$.</td>
</tr>
<tr>
<td>$\leq_S$</td>
<td>The squashed order: The largest element in $A + B$ is in $B$, or $A = B$.</td>
</tr>
<tr>
<td>$\leq_A$</td>
<td>The antilexicographic order: The largest element in $A + B$ is in $A$, or $A = B$.</td>
</tr>
<tr>
<td>$a \leq_L b$</td>
<td>The lexicographic order defined for multisets. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two elements of $S(k_1, k_2, \ldots, k_n)$. Then $a \leq_L b$ if $a_1 &lt; b_1$, or if $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i &lt; b_i$ for some $i$, $1 &lt; i \leq n$, or if $a = b$.</td>
</tr>
</tbody>
</table>
# Appendix A. Table of Symbols

A, B denote subsets of \([n]\).

\(B, C\) denote collections of subsets of \([n]\) in squashed order.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A \subsetneq B \text{ or } B \supsetneq A)</td>
<td>(A \subseteq B) and (A \neq B).</td>
</tr>
<tr>
<td>(A \subsetneq B \text{ or } B \supsetneq A)</td>
<td>(A \subsetneq B) for all (B \in B).</td>
</tr>
<tr>
<td>(A \supsetneq B \text{ or } B \subsetneq A)</td>
<td>(A \supsetneq B) for all (B \in B).</td>
</tr>
<tr>
<td>(B \subsetneq C \text{ or } C \supsetneq B)</td>
<td>(B \subsetneq C) for all (B \in B) and (C \in C).</td>
</tr>
<tr>
<td>(F_{n,k}(p))</td>
<td>The first (p) consecutive (k)-subsets of ([n]) in squashed order.</td>
</tr>
<tr>
<td>(C_{n,k}(p))</td>
<td>A collection of (p) consecutive (k)-subsets of ([n]) in squashed order.</td>
</tr>
<tr>
<td>(L_{n,k}(p))</td>
<td>The last (p) consecutive (k)-subsets of ([n]) in squashed order.</td>
</tr>
<tr>
<td>(N^m_{n,k}(p))</td>
<td>The collection of the (p) consecutive (k)-subsets of ([n]) in squashed order which come immediately after (F_{n,k}(m)) in squashed order.</td>
</tr>
<tr>
<td>(P^m_{n,k}(p))</td>
<td>The collection of the (p) consecutive (k)-subsets of ([n]) in squashed order which come immediately before (L_{n,k}(m)) in squashed order.</td>
</tr>
<tr>
<td>(F_k(p), C_k(p), L_k(p), N^m_k(p), P^m_k(p))</td>
<td>Used in lieu of (F_{n,k}(p), C_{n,k}(p), L_{n,k}(p), N^m_{n,k}(p),) and (P^m_{n,k}(p)) when the universal set is implicitly known or when it need not be known.</td>
</tr>
<tr>
<td>(F(p, B))</td>
<td>The first (p) sets of (B) in squashed order.</td>
</tr>
<tr>
<td>(L(p, B))</td>
<td>The last (p) sets of (B) in squashed order.</td>
</tr>
</tbody>
</table>

---

Table A.1: Notation: 3
\( B \) denotes a \( k \)-subset of \([n]\). \( \mathcal{B} \) denotes a collection of \( k \)-subsets of \([n]\).

\( C \) denotes a collection of subsets of \([n]\).

\( \mathcal{M} \) denotes a collection of \( k \)-vectors of \( S(k_1, k_2, \ldots, k_n) \).

\[
\begin{align*}
\Delta B & \quad \{ D : D \subset B, |D| = k - 1 \}. \\
\nabla B & \quad \{ D \subseteq [n] : D \supset B, |D| = k + 1 \}. \\
\Delta_N B & \quad \{ D : D \in \Delta B, D \notin \Delta C \text{ for all } C \subsetneq B \}. \\
\nabla_N B & \quad \{ D : D \in \nabla B, D \notin \nabla C \text{ for all } C \supsetneq B \}. \\
\Delta B & \quad \bigcup_{\mathcal{B} \in \mathcal{B}} \Delta B. \\
\nabla B & \quad \bigcup_{\mathcal{B} \in \mathcal{B}} \nabla B. \\
\Delta_N B & \quad \bigcup_{\mathcal{B} \in \mathcal{B}} \Delta_N B. \\
\nabla_N B & \quad \bigcup_{\mathcal{B} \in \mathcal{B}} \nabla_N B. \\
\Delta^{(i)} B & \quad \{ D : D \subset B, |D| = l \}. \\
\nabla^{(l)} B & \quad \{ D \subseteq [n] : D \supset B, |D| = l \}. \\
\Delta^{(i)} C & \quad \bigcup_{\mathcal{C} \in \mathcal{C}} \Delta^{(i)} C. \\
\nabla^{(i)} C & \quad \bigcup_{\mathcal{C} \in \mathcal{C}} \nabla^{(i)} C. \\
\nabla^{||} C & \quad \bigcup_{\mathcal{C} \in \mathcal{C}} \nabla^{||} C. \\
\nabla^{(l)} C & \quad \bigcup_{\mathcal{C} \in \mathcal{C}} \nabla^{(l)} C. \\
\nabla^{(||)} C & \quad \bigcup_{\mathcal{C} \in \mathcal{C}} \nabla^{(||)} C. \\
\n\bigdiamond^{(i)} C & \quad \bigcup_{\mathcal{C} \in \mathcal{C}} \bigdiamond^{(i)} C. \\
\bigdiamond^{(||)} C & \quad \bigcup_{\mathcal{C} \in \mathcal{C}} \bigdiamond^{(||)} C. \\
\Delta \mathcal{M} & \quad \{ \mathbf{m} = (m_1, m_2, \ldots, m_n) : |\mathbf{m}| = k - 1; \}
\quad \{ m_1, \ldots, m_{i-1}, m_i + 1, \ldots, m_n \} \in \mathcal{M} \text{ for some } i, 1 \leq i \leq n \}.
\end{align*}
\]
Appendix B

Table of Definitions
A, B denote subsets of \([n]\). \(\mathcal{A}, \mathcal{B}\) denote collections of subsets of \([n]\).

**antichain**
\(\mathcal{A}\) is an antichain if for any \(A, B \in \mathcal{A}\), \(A \nsubseteq B\).

**antilexicographic antichain**
Let \(\mathcal{A}\) be an antichain on \([n]\) with largest and smallest set size \(h\) and \(l\) respectively. \(\mathcal{A}\) is antilexicographic if, for \(i = l, l+1, \ldots, h\), the \(i\)-sets in \(\triangledown(i)\mathcal{A}\) come after the sets in \(\mathcal{A}\) in squashed order so that the sets in \(\triangledown(i)\mathcal{A} = \mathcal{A}^{(i)} \cup \triangledown(i)\mathcal{A}\) form an initial segment of the \(i\)-sets in antilexicographic order.

**antilexicographic order, \(\leq_{A}\)**
\(A \leq_{A} B\) if the largest element in \(A+B\) is in \(A\) or if \(A=B\).

**average set size of \(B\), \(\overline{B}\)**
\(\overline{B}\) is the average set size of \(B\) if every unordered pair \((i, j) \in [n] \times [n]\) there exists \(C \in \mathcal{C}\) such that \(i \in C\) and \(j \notin C\).

**\(k\)-binomial representation**
\(\binom{n}{k}\) is \(\binom{n}{k}\) if \(i \in C\) and \(j \notin C\).

**complement of \(B\), \(B'\)**
\([n] \setminus B\).

**complement of \(B\), \(B'\)**
\(\{B : B' \in B\}\). If the sets in \(B = \{B_1, B_2, \ldots, B_m\}\) are ordered, then \(B' = \{B_1', B_2', \ldots, B_m'\}\).

**completely separating system**
\(C\) is a completely separating system on \([n]\) if for each unordered pair \((i, j) \in [n] \times [n]\) there exists \(C \in \mathcal{C}\) such that \(i \in C\) and \(j \notin C\).

**convex function**
\(f : D \to \mathbb{R}\) is convex if \(f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b)\) for all \(a, b \in D\) and all \(\lambda \in \mathbb{R}^{k} \cup \{0\}\) such that \(\lambda a + (1 - \lambda)b \in D\) and \(0 \leq \lambda \leq 1\).

**correspond**
\(\mathcal{A}\) and \(\mathcal{B}\) correspond if \(|\mathcal{A}| = |\mathcal{B}|\), \(|\Delta_{\mathcal{N}}\mathcal{A}| = |\Delta_{\mathcal{N}}\mathcal{B}|\), and \(|\triangledown_{\mathcal{N}}\mathcal{A}| = |\triangledown_{\mathcal{N}}\mathcal{B}|\).

**correspondence**
There is a correspondence between \(\mathcal{A}\) and \(\mathcal{B}\) if \(\mathcal{A}\) and \(\mathcal{B}\) correspond.

**dual of \(B\)**
Let \(\mathcal{B} = \{B_1, \ldots, B_m\}\). The dual of \(\mathcal{B}\) is \(\mathcal{C} = \{X_1, \ldots, X_n\}\) with \(X_i = \{j : i \in B_j, 1 \leq j \leq m\}\) for each \(i \in [n]\).

**filter of \(B\), \(\mathcal{F}B\)**
\(\{D \subseteq [n] : D \supseteq B, B \in B\}\).

**filter of \(B\) on level \(k\), \(\mathcal{F}^{(k)}B\)**
\(\{D : D \in \mathcal{F}B, |D| = k\}\).

**flat**
\(\mathcal{B}\) is flat if it consists of sets on at most two consecutive levels.

**flat counterpart of \(B\)**
A flat collection of sets of size \(|\mathcal{B}|\) and volume \(V(\mathcal{B})\).

**full**
Let \(\mathcal{A}\) be a squashed antichain on \([n]\) with smallest set size \(l\). \(\mathcal{A}\) is full if, either \(\mathcal{A}_l = L_{n,l}[\mathcal{A}^{(l)}]\), or \(\mathcal{A} = F_{n,1}[\mathcal{A}]\) and \(|\triangledown^{(l-1)}\mathcal{A}| = \binom{n}{l}\).

---

**Table B.1: Definitions:**

1. **antichain**
2. **antilexicographic antichain**
3. **antilexicographic order, \(\leq_{A}\)**
4. **average set size of \(B\), \(\overline{B}\)**
5. **\(k\)-binomial representation**
6. **complement of \(B\), \(B'\)**
7. **completely separating system**
8. **convex function**
9. **correspond**
10. **correspondence**
11. **dual of \(B\)**
12. **filter of \(B\), \(\mathcal{F}B\)**
13. **filter of \(B\) on level \(k\), \(\mathcal{F}^{(k)}B\)**
14. **flat**
15. **flat counterpart of \(B\)**
16. **full**
Appendix B. Table of Definitions

$B$ denotes a $k$-subset of $[n]$. $\mathcal{B}$ denotes a collection of subsets of $[n]$.

<table>
<thead>
<tr>
<th>Definition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ideal of $B$, $\mathcal{IB}$</td>
<td>${D : D \subseteq B, B \in \mathcal{B}}$.</td>
</tr>
<tr>
<td>ideal of $B$ on level $k$, $\mathcal{T}^{(k)}B$</td>
<td>${D : D \in \mathcal{IB},</td>
</tr>
<tr>
<td>$l$-image of $B$, $\mathcal{S}^{(l)}B$</td>
<td>$\triangle^{(l)}B \cup \mathcal{B}^{(l)} \cup \nabla^{(l)}_N B$.</td>
</tr>
<tr>
<td>initial segment</td>
<td>$B$ is an initial segment of $k$-sets in squashed order if $B = F_{n,k}([B])$ or $B$ is an initial segment of $k$-subsets of $[n]$ in antilexicographic order if $B = L_{n,k}([B])$.</td>
</tr>
<tr>
<td>level of $B$</td>
<td>The integers $i$, $0 \leq i \leq n$, designating the size of the sets in $B$.</td>
</tr>
<tr>
<td>lexicographic order, $\leq_L$</td>
<td>Defined for multisets. Let $S(k_1, k_2, \ldots, k_n)$ be a multiset and let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two vectors of $S(k_1, k_2, \ldots, k_n)$. Then $a \leq_L b$ if $a_1 &lt; b_1$, or if $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}, a_i &lt; b_i$ for some $i$, $1 &lt; i \leq n$, or $a = b$.</td>
</tr>
<tr>
<td>maximum volume in $\Lambda_{n,\delta}$</td>
<td>$\mathcal{A} \in \Lambda_{n,\delta}$ achieves maximum volume in $\Lambda_{n,\delta}$ if $V(\mathcal{A}) = \max_{\mathcal{A} \in \Lambda_{n,\delta}} V(\mathcal{B})$.</td>
</tr>
<tr>
<td>minimal completely separating system on $[n]$</td>
<td>A completely separating system on $[n]$ with smallest size.</td>
</tr>
<tr>
<td>minimum volume (size ideal) in $\Lambda_{n,\delta}$</td>
<td>$\mathcal{A} \in \Lambda_{n,\delta}$ achieves minimum volume (minimum size ideal) in $\Lambda_{n,\delta}$ if $V(\mathcal{A}) = \min_{\mathcal{A} \in \Lambda_{n,\delta}} V(\mathcal{B})$ $(</td>
</tr>
<tr>
<td>multiset on $[n]$</td>
<td>Let $k_1, \ldots, k_n \in \mathcal{N}$. The multiset $S(k_1, k_2, \ldots, k_n)$ consists of the collections of elements of $[n]$ which contain at most $k_i$ occurrences of the element $n - i + 1$, $i \in [n]$.</td>
</tr>
<tr>
<td>new-shade of $B$, $\nabla_N B$</td>
<td>${D : D \in \nabla B, D \notin \nabla C$ for all $C \geq B}$.</td>
</tr>
<tr>
<td>new-shade on level $l$ of $B$, $\nabla^{(l)}_N B$</td>
<td>${D : D \in \nabla^{(l)} B, D \notin \nabla^{(l)} C$ for all $C \geq B}$.</td>
</tr>
<tr>
<td>new-shade of a collection of $k$-sets $\mathcal{B}$, $\nabla_N \mathcal{B}$</td>
<td>$\cup_{\mathcal{B} \in \mathcal{B}} \nabla_N B$.</td>
</tr>
<tr>
<td>new-shade on level $l$ of $\mathcal{B}$, $\nabla^{(l)}_N \mathcal{B}$</td>
<td>$\cup_{\mathcal{B} \in \mathcal{B}} \nabla^{(l)}_N B$.</td>
</tr>
<tr>
<td>new-shadow of $B$, $\triangle_N B$</td>
<td>${D : D \in \triangle B, D \notin \triangle C$ for all $C \geq B}$.</td>
</tr>
<tr>
<td>new-shadow on level $l$ of $B$, $\triangle^{(l)}_N B$</td>
<td>${D : D \in \triangle^{(l)} B, D \notin \triangle^{(l)} C$ for all $C \geq B}$.</td>
</tr>
<tr>
<td>new-shadow of a collection of $k$-sets $\mathcal{B}$, $\triangle_N \mathcal{B}$</td>
<td>$\cup_{\mathcal{B} \in \mathcal{B}} \triangle_N B$.</td>
</tr>
<tr>
<td>new-shadow on level $l$ of $\mathcal{B}$, $\triangle^{(l)}_N \mathcal{B}$</td>
<td>$\cup_{\mathcal{B} \in \mathcal{B}} \triangle^{(l)}_N B$.</td>
</tr>
</tbody>
</table>

Table B.1: Definitions: 2
\( B \) denotes a collection of subsets of \([n]\).

<table>
<thead>
<tr>
<th>Definition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>non-reducible</td>
<td>The negation of reducible.</td>
</tr>
<tr>
<td>parameters of ( B ), ( p_i )</td>
<td>The number of members of ( B ) of size ( i ).</td>
</tr>
<tr>
<td>partition of ( B )</td>
<td>A collection of pairwise disjoint collections of subsets of ( B ) whose union is ( B ).</td>
</tr>
<tr>
<td>profile of ( B ), ((p_0, \ldots, p_n))</td>
<td>The ((n+1)) tuple of the parameters of ( B ).</td>
</tr>
<tr>
<td>profile-unique</td>
<td>Let ( P ) be some property of antichains and let ( A ) be an antichain with property ( P ). If ( A \equiv B ) for each antichain ( B ) with property ( P ), then we say that ( A ) is a profile-unique antichain with property ( P ).</td>
</tr>
<tr>
<td>profile-equivalent, ( \equiv )</td>
<td>Let ( A ) and ( B ) be two antichains. ( A \equiv B ) if ( A ) and ( B ) have the same profile.</td>
</tr>
<tr>
<td>( l )-projection of ( B ), ( \triangle(l)B )</td>
<td>( \triangle(l)B \cup B^{(l)} \cup \coarsening(l)B ).</td>
</tr>
<tr>
<td>rank</td>
<td>Applies to an element of a multiset. Let ( \mathbf{m} = (m_1, m_2, \ldots, m_n) ) be a vector of the multiset ( S(k_1, k_2, \ldots, k_n) ). The rank of ( \mathbf{m} ) is (</td>
</tr>
<tr>
<td>real ( k )-binomial representation</td>
<td>Let ( p, k \in \mathbb{Z}^+ ). Then ( p = \binom{k}{\ell}, x \in \mathbb{R}^+, x \geq k ), is the real ( k )-binomial representation of ( p ).</td>
</tr>
<tr>
<td>reducible</td>
<td>Let ( A ) be a squashed antichain on ([n]) with largest set size ( h ). ( A ) is reducible if there exists ( t \in \mathbb{Z}^+ ), ( 0 &lt; t \leq</td>
</tr>
<tr>
<td>( k )-set, ( k )-subset or ( k )-superset</td>
<td>Set of size ( k ).</td>
</tr>
<tr>
<td>set on level ( k )</td>
<td>Set of size ( k ).</td>
</tr>
</tbody>
</table>
### Appendix B. Table of Definitions

A. \( B \) denote subsets of \([n]\). \( \mathcal{B} \) denotes a collection of subsets of \([n]\).

<table>
<thead>
<tr>
<th>Definition</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shade of a ( k )-set ( B ), ( \nabla B )</td>
<td>( { D \subseteq [n] : D \supset B,</td>
</tr>
<tr>
<td>Shade on level ( l ) of a set ( B ), ( \nabla^{(l)} B )</td>
<td>( \bigcup_{B \in \mathcal{B}} \nabla^{(l)} B )</td>
</tr>
<tr>
<td>Shade of a collection of ( k )-sets ( \mathcal{B} ), ( \nabla \mathcal{B} )</td>
<td>( { D : D \subseteq B,</td>
</tr>
<tr>
<td>Shade on level ( l ) of ( \mathcal{B} ), ( \nabla^{(l)} \mathcal{B} )</td>
<td>( \bigcup_{\mathcal{B} \in \mathcal{B}} \nabla^{(l)} \mathcal{B} )</td>
</tr>
<tr>
<td>Shadow of a ( k )-set ( B ), ( \triangle B )</td>
<td>( { D : D \subset B,</td>
</tr>
<tr>
<td>Shadow on level ( l ) of a set ( B ), ( \triangle^{(l)} B )</td>
<td>( \bigcup_{B \in \mathcal{B}} \triangle^{(l)} B )</td>
</tr>
<tr>
<td>Shadow of a collection of ( k )-sets ( \mathcal{B} ), ( \triangle \mathcal{B} )</td>
<td>( { D : D \subset B,</td>
</tr>
<tr>
<td>Shadow on level ( l ) of ( \mathcal{B} ), ( \triangle^{(l)} \mathcal{B} )</td>
<td>( \bigcup_{\mathcal{B} \in \mathcal{B}} \triangle^{(l)} \mathcal{B} )</td>
</tr>
<tr>
<td>Shadow of a collection of vectors ( \mathcal{M} ), ( \triangle \mathcal{M} )</td>
<td>Let ( \mathcal{M} ) be a collection of ( k )-vectors of ( S(k_1, k_2, \ldots, k_n) ). ( \triangle \mathcal{M} = { m = (m_1, m_2, \ldots, m_n) :</td>
</tr>
<tr>
<td>Size or cardinality of ( B ), (</td>
<td>B</td>
</tr>
<tr>
<td>Squashed antichain</td>
<td>Let ( \mathcal{A} ) be an antichain on ([n]) with largest and smallest set size ( h ) and ( l ) respectively. ( \mathcal{A} ) is squashed if, for ( i = h, h - 1, \ldots, l ), the ( i )-sets in ( \triangle^{(i)} \mathcal{A} ) come before the sets in ( \mathcal{A}^{(i)} ) in squashed order so that the sets in ( T^{(i)} \mathcal{A} = \triangle^{(i)} \mathcal{A} \cup \mathcal{A}^{(i)} ) form an initial segment of the ( i )-sets in squashed order. ( A \leq_{\leq_{B}} B ) if the largest element in ( A + B ) is in ( B ), or if ( A = B ). ( (A \setminus B) \cup (B \setminus A) ).</td>
</tr>
<tr>
<td>Squashed order, ( \leq_{B} )</td>
<td>( \mathcal{B} ) is a terminal segment of ( k )-subsets of ([n]) in squashed order if ( \mathcal{B} = L_{n,k}([B]) ) or ( \mathcal{B} ) is a terminal segment of ( k )-sets in antilexicographic order if ( \mathcal{B} = F_{n,k}([B]) ). An element of a multiset ( S(k_1, k_2, \ldots, k_n) ). The vector ( m \in S(k_1, k_2, \ldots, k_n) ) is denoted by ( m = (m_1, m_2, \ldots, m_n) ), ( m_i \leq k_i ), where ( m_i ) denotes the number of occurrences of the element ( n - i + 1 ) in ( m ). A vector of rank ( k ). ( \sum_{B \in \mathcal{B}}</td>
</tr>
<tr>
<td>Symmetric difference of ( A ) and ( B ), ( A + B )</td>
<td>Let ( \mathcal{A}, \mathcal{B} \in \Lambda_{n,k} ). Then ( \mathcal{A} ) is volume-equivalent to ( \mathcal{B} ) if ( V(\mathcal{A}) = V(\mathcal{B}) ).</td>
</tr>
<tr>
<td>Terminal segment</td>
<td></td>
</tr>
<tr>
<td>Vector</td>
<td></td>
</tr>
<tr>
<td>( k )-vector</td>
<td></td>
</tr>
<tr>
<td>Volume of ( \mathcal{B} ), ( V(\mathcal{B}) )</td>
<td>( \sum_{B \in \mathcal{B}}</td>
</tr>
<tr>
<td>Volume-equivalent, ( \equiv_{V(\Lambda_{n,k})} )</td>
<td>Let ( \mathcal{A}, \mathcal{B} \in \Lambda_{n,k} ). Then ( \mathcal{A} ) is volume-equivalent to ( \mathcal{B} ) if ( V(\mathcal{A}) = V(\mathcal{B}) ).</td>
</tr>
</tbody>
</table>

Table B.1: Definitions: 4
Appendix C

Naming and Labelling for Some Results
<table>
<thead>
<tr>
<th>Result</th>
<th>Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sperner’s lemma</td>
<td>Lemma 2.33</td>
</tr>
<tr>
<td>the LYM inequality</td>
<td>Theorem 2.75</td>
</tr>
<tr>
<td>the flat antichain conjecture</td>
<td>Conjecture 4.1</td>
</tr>
<tr>
<td>the 3-levels result</td>
<td>Theorem 7.1</td>
</tr>
</tbody>
</table>

Table C.1: Naming and labelling for some results
Appendix D

The Algorithm Used to Show that the 3-Levels Result Holds for $n \leq 32$
Appendix D. The Algorithm Used in the Proof of the 3-Levels Result

This appendix presents Algorithm D.1 which is the algorithm used to show that Theorem 7.1 holds for values of $n \leq 32$ (see Proposition 7.14). Algorithm D.1 computes the size of the new-shadow and the size of the new-shade of a set. It is given and proved in Section D.1, and an implementation is presented and discussed in Section D.2. The program code and its output for values of $n \leq 32$ are found Page 217 and Page 221 respectively.

### D.1 Computing the Size of the New-Shadow and the Size of the New-Shade of the $p$th Set in Squashed Order

Let $B$ be the $p$th $k$-subset of $[n]$ in squashed order, $k \in \mathbb{Z}^+$, $1 \leq p \leq \binom{n}{k}$. Algorithm D.1 below is used to compute $|\Delta_N B|$ and $|\nabla_N B|$. We present Algorithm D.1 first, before describing it briefly, and proving it is correct.

**Algorithm D.1.**

Step 1. set $p = 1$;
   set $I(1) = k$;
   set $\sigma(0) = (0, \ldots, 0)$;

Step 2. if $\sigma(p-1)[k] = n - k + 1$ then stop;

Step 3. set $Z(p) = \sigma(p-1)[1]$;
   if $I(p) = 0$ then
     set $i = 1$;
     while $\sigma(p-1)[i] = \sigma(p-1)[i+1]$ and $i < k$
     do $i = i + 1$;
     set $I(p+1) = i - 1$;
     set $\begin{cases} 
       \sigma(p)[i] = 0; & \text{for } i = 1, \ldots, I(p+1), \\
       \sigma(p)[i] = \sigma(p-1)[i]+1; & \text{for } i = I(p+1)+1, \\
       \sigma(p)[i] = \sigma(p-1)[i]; \quad & \text{for } i = I(p+1)+2, \ldots, k;
     \end{cases}$
   else
     set $I(p+1) = I(p) - 1$;
     set $\begin{cases} 
       \sigma(p)[i] = \sigma(p-1)[i]; & \text{for } i = 1, \ldots, k \text{ and } i \neq I(p+1)+1, \\
       \sigma(p)[i] = \sigma(p-1)[i]+1; & \text{for } i = I(p+1)+1;
     \end{cases}$

Step 4. set $p = p + 1$;
repeat Step 2.
Appendix D. The Algorithm Used in the Proof of the 3-Levels Result

\( p \) is the counter which gives the position of \( B \) in the squashed order. \( I(p) \) is an integer whose value is \( |\triangle_N B| \) as shown by Lemma D.7, and \( Z(p) \) is an integer whose value is \( |\nabla_N B| \) as shown by Lemma D.8.

\( \sigma(p) \) is an ordered \( k \)-tuple whose \( i \)th element is denoted by \( \sigma(p)[i] \). \( \sigma(p) \) is used to store a representation of the \( k \)-binomial representation of \( p \) as explained here: If 
\[ p = \sum_{i=1}^{k} \binom{a_i}{i} \]

is the \( k \)-binomial representation of \( p \), then the \( i \)th element of \( \sigma(p) \) is \( a_i - i + 1 \) for \( i \geq t \) and 0 for \( i < t \). That is, \( \sigma(p)[i] = a_i - i + 1 \) for \( i = t, \ldots, k \), and \( \sigma(p)[i] = 0 \) for \( i = 1, \ldots, t - 1 \). This is shown in Lemma D.5.

Step 1 in Algorithm D.1 is the initialisation step whereby \( p = 1 \), \( I(1) = k \), and \( \sigma(0) \) is the zero \( k \)-tuple. By Corollary 2.41, it can be seen that Lemma D.7’s claim that \( I(1) = |\triangle_N B| = |\triangle_N F_k(1)| \) is valid. Step 2 ensures that the algorithm terminates properly. This is shown in Lemma D.6. Step 3 sets \( Z(p) \). For \( p = 1 \), \( Z(p) = 0 \) and, by Lemma 2.60(ii), Lemma D.8’s claim that \( Z(1) = |\nabla_N B| = |\nabla_N F_k(1)| \) is valid. Step 3 also sets \( \sigma(p) \) so that \( \sigma(p) \) is the representation, as given by Lemma D.5, of the \( k \)-binomial representation of \( p \).

When \( p = 1 \), \( I(1) \neq 0 \), and after Step 3, \( \sigma(1)[k] = 1 \) and \( \sigma(1)[i] = 0 \) for \( i = 1, \ldots, k - 1 \). The \( k \)-binomial representation of \( p = 1 \) is \( \binom{k}{1} \). Therefore Lemma D.5’s claim that \( \sigma(1)[k] = k - k + 1 \) and \( \sigma(1)[i] = 0 \) for \( i = 1, \ldots, k - 1 \) is verified. Note that \( I(2) = k - 1 \) at this stage, whence, by Corollary 2.41, Lemma D.7’s claim that \( I(2) \) is the size of the new-shadow of the second \( k \)-set in squashed order is valid. Finally, Step 5 increments \( p \) before Step 2 is repeated.

Table D.1 lists \( p \), \( I(p) \), \( Z(p) \), and \( \sigma(p) \) for values of \( p \) between 1 and \( \binom{4}{4} \). We give a few examples of how \( \sigma(p) \) is a representation of the \( k \)-binomial representation of \( p \).

Example D.2. Let \( n = 7 \) and \( k = 4 \). Let \( p = 3 \). Then the 4-binomial representation of \( p \) is \( \binom{1}{4} + \binom{3}{3} + \binom{1}{1} \). This corresponds to \( \sigma(3) = (0, 1, 1, 1) \).

Let \( p = 12 \). Then the 4-binomial representation of \( p \) is \( \binom{5}{4} + \binom{3}{3} + \binom{2}{2} \). This corresponds to \( \sigma(12) = (0, 2, 2, 2) \).

Let \( p = 23 \). Then the 4-binomial representation of \( p \) is \( \binom{6}{4} + \binom{3}{3} + \binom{3}{2} + \binom{1}{1} \). This corresponds to \( \sigma(23) = (1, 2, 2, 3) \).
<table>
<thead>
<tr>
<th>$p$</th>
<th>$I(p)$</th>
<th>$Z(p)$</th>
<th>$\sigma(p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>(0, 0, 0, 1)</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0</td>
<td>(0, 0, 1, 1)</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0</td>
<td>(0, 1, 1, 1)</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>(1, 1, 1, 1)</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1</td>
<td>(0, 0, 0, 2)</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>0</td>
<td>(0, 0, 1, 2)</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>0</td>
<td>(0, 1, 1, 2)</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0</td>
<td>(1, 1, 1, 2)</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>1</td>
<td>(0, 0, 2, 2)</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>0</td>
<td>(0, 1, 2, 2)</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>0</td>
<td>(1, 1, 2, 2)</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>1</td>
<td>(0, 2, 2, 2)</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
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</tr>
<tr>
<td>14</td>
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<td>1</td>
<td>(2, 2, 2, 2)</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>2</td>
<td>(0, 0, 0, 3)</td>
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<tr>
<td>16</td>
<td>3</td>
<td>0</td>
<td>(0, 0, 1, 3)</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td>0</td>
<td>(0, 1, 1, 3)</td>
</tr>
<tr>
<td>18</td>
<td>1</td>
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<td>0</td>
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</tr>
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<td>1</td>
<td>(0, 2, 2, 3)</td>
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<td>0</td>
<td>(1, 2, 2, 3)</td>
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<td>1</td>
<td>(2, 2, 2, 3)</td>
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<td>2</td>
<td>(0, 0, 3, 3)</td>
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<td>(0, 1, 3, 3)</td>
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<td>(1, 1, 3, 3)</td>
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<td>(3, 3, 3, 3)</td>
</tr>
<tr>
<td>35</td>
<td>0</td>
<td>3</td>
<td>(0, 0, 0, 4)</td>
</tr>
</tbody>
</table>

Table D.1: An example run for Algorithm D.1: $n = 7, k = 4$
Appendix D. The Algorithm Used in the Proof of the 3-Levels Result

We now prove that Algorithm D.1 computes $|\triangle_X B|$ and $|\nabla_X B|$. This requires Lemmas D.3 to D.8.

**Lemma D.3.** Let $k \in \mathbb{Z}^+$ and $1 \leq p \leq \binom{n}{k}$. In Algorithm D.1, $\sigma(p)[i] \geq 0$ for $i = 1, \ldots, k$.

**Proof.** The proof uses induction on $p$. For $p = 1$, $\sigma(1)$ is initialised to the zero $k$-tuple. Assume that Lemma D.3 holds for $p - 1$. In Step 3, $\sigma(p)[i]$ is either set to 0, or to $\sigma(p-1)[i]$, or to $\sigma(p-1)[i] + 1$. As $\sigma(p-1)[i] \geq 0$ for $i = 1, \ldots, k$ by the induction hypothesis, it follows that $\sigma(p)[i] \geq 0$ for $i = 1, \ldots, k$.

**Lemma D.4.** Let $k \in \mathbb{Z}^+$. Let $p$ be given such that $1 \leq p \leq \binom{n}{k}$. Then the smallest $i$ for which $\sigma(p)[i] \neq 0$ is $I(p+1) + 1$, and the largest $i$ for which $\sigma(p)[i] = 0$ is $I(p)$. We consider two cases.

(i) Assume that $I(p) = 0$. Then $\sigma(p)[i] = 0$ for $i = 1, \ldots, I(p+1)$, and $\sigma(p)[I(p+1) + 1] = \sigma(p-1)[I(p+1) + 1] + 1$. Hence $\sigma(p)[I(p+1) + 1] > 0$ as $\sigma(p-1)[I(p+1) + 1] \geq 0$ by Lemma D.3, thus $I(p+1) + 1$ is the smallest $i$ for which $\sigma(p)[i] \neq 0$. By the induction hypothesis, $\sigma(p-1)[i] > 0$ for $i > I(p)$, that is, for $i = 1, \ldots, k$ as $I(p) = 0$. Therefore, $\sigma(p)[i] > 0$ for $i = I(p+1) + 2, \ldots, k$, as $\sigma(p)[i] = \sigma(p-1)[i]$ for these values of $i$, and $I(p+1)$ is the largest $i$ for which $\sigma(p)[i] = 0$.

(ii) Assume that $I(p-1) \neq 0$. By the induction hypothesis, $\sigma(p-1)[i] = 0$ for $i = 1, \ldots, I(p)$ and $\sigma(p-1)[i] > 0$ for $i = I(p) + 1, \ldots, k$. As $\sigma(p)[i] = \sigma(p-1)[i]$ for $i \neq I(p+1) + 1$, that is, for $i \neq I(p)$, it follows that $\sigma(p)[i] = 0$ for $i = 1, \ldots, I(p)-1$, that is, for $i = 1, \ldots, I(p+1)$, and that $\sigma(p)[i] > 0$ for $i = I(p) + 1, \ldots, k$, that is, for $i = I(p+1) + 2, \ldots, k$. Finally, $\sigma(p)[I(p+1) + 1] > 0$ as $\sigma(p)[I(p+1) + 1] = \ldots$
$$\sigma(p-1)[I(p+1)] + 1$$ and $$\sigma(p-1)[I(p+1) + 1] \geq 0$$ by Lemma D.3. This concludes the proof. \[ \square \]

**Lemma D.5.** Let $$k \in \mathbb{Z}^+.$$ Let $$p$$ be given such that $$1 \leq p \leq \left(\frac{n}{k}\right)$$ and let $$p = \sum_{i=1}^{k} \left(\begin{array}{c}a_i \\
i\end{array}\right)$$ be the $$k$$-binomial representation of $$p.$$ Then $$\sigma(p)[i] = a_i - i + 1$$ for $$i = t, \ldots, k,$$ and $$\sigma(p)[i] = 0$$ for $$i = 1, \ldots, t - 1.$$ That is, $$\sigma(p)$$ is a representation of the $$k$$-binomial representation of $$p.$$

**Proof.** The proof uses induction on $$p.$$ Let $$p = 1.$$ The $$k$$-binomial representation of 1 is $$\left(\begin{array}{c}1 \\
k\end{array}\right).$$ Also, $$\sigma(1)[i] = 0$$ for $$i = 1, \ldots, k - 1$$ and $$\sigma(1)[k] = 1.$$ Thus Lemma D.5 holds for $$p = 1.$$ Assume that Lemma D.5 holds for $$p - 1$$ and let $$\sum_{i=1}^{k} \left(\begin{array}{c}b_i \\
k\end{array}\right)$$ the $$k$$-binomial representation of $$p - 1.$$ By the induction hypothesis, it follows that $$\sigma(p-1)[i] = b_i - i + 1$$ for $$i \geq t,$$ and $$\sigma(p-1)[i] = 0$$ for $$i < t.$$ We consider two cases.

(i) Assume that $$I(p) = 0.$$ Then $$\sigma(p-1)[i] > 0$$ for $$i = 1, \ldots, k$$ by Lemma D.4. By the induction hypothesis, $$\sigma(p-1)[i] = b_i - i + 1$$ for $$i = t, \ldots, k,$$ that is $$\sigma(p-1)[i] > 0$$ for $$i = t, \ldots, k.$$ It follows that $$t = 1.$$ The condition $$\sigma(p-1)[i] = \sigma(p-1)[i+1]$$ is equivalently stated as $$b_i - i + 1 = b_{i+1} - i,$$ that is, $$b_i + 1 = b_{i+1}.$$ If follows that, at exit of the iteration controlled by the condition $$\sigma(p-1)[i] = \sigma(p-1)[i+1],$$ $$I(p + 1) + 1$$ is the smallest integer $$i$$ for which $$\sigma(p-1)[i] \neq \sigma(p-1)[i+1],$$ that is, $$b_i + 1 \neq b_{i+1}.$$ It follows, as $$t = 1,$$ that the $$k$$-binomial representation of $$p$$ is $$\left(\begin{array}{c}\hat{b}_1(I(p+1)+1) \\
I(p+1)+1\end{array}\right) + \sum_{i=I(p+1)+2}^{k} \left(\begin{array}{c}b_i \\
k\end{array}\right)$$ by Lemma 2.13. Now, $$\sigma(p)[i] = 0$$ for $$i = 1, \ldots, I(p+1),$$ $$\sigma(p)[I(p+1) + 1] = \sigma(p-1)[I(p+1) + 1],$$ and $$\sigma(p)[i] = \sigma(p-1)[i]$$ for $$i = I(p+2) + 2, \ldots, k.$$ Thus, applying the induction hypothesis again, $$\sigma(p)[I(p+1) + 1] = [\hat{b}_1(I(p+1)+1) - (I(p+1) + 1) + 1]$$ and $$\sigma(p)[i] = b_i - i + 1$$ for $$i = I(p+2) + 2, \ldots, k.$$ The result follows.

(ii) Assume that $$I(p) \neq 0.$$ By Lemma D.4 $$\sigma(p-1)[i] = 0$$ for $$i = 1, \ldots, I(p).$$ Thus, by the induction hypothesis, the $$k$$-binomial representation of $$p - 1$$ is $$\sum_{i=I(p)+1}^{k} \left(\sigma(p-1)[i]+i-1\right) = \sum_{i=I(p)+1}^{k} \left(\begin{array}{c}b_i \\
k\end{array}\right).$$ By Lemma 2.14, the $$k$$-binomial representation of $$p$$ is $$\left(\begin{array}{c}\hat{b}_1(I(p)) \\
I(p)\end{array}\right) + \sum_{i=I(p)+1}^{k} \left(\begin{array}{c}b_i \\
k\end{array}\right).$$ When $$I(p) \neq 0,$$ $$\sigma(p)[i] = \sigma(p-1)[i]$$ for $$i \neq I(p)+1,$$ that is, for $$i \neq I(p),$$ and $$\sigma(p)[I(p)+1] = \sigma(p-1)[I(p)+1] + 1,$$ that is, $$\sigma(p)[I(p)] = \sigma(p-1)[I(p)] = 1$$ since $$\sigma(p-1)[I(p)] = 0.$$ The result follows. \[ \square \]
Lemma D.6. Algorithm D.1 terminates properly.

Proof. Let \( p = \binom{n}{k} \). Then \( \sigma(p)[k] = n - k + 1 \) by Lemma D.5. Therefore, when \( p = \binom{n}{k} + 1 \) in Step 2, \( \sigma(p - 1)[k] = n - k + 1 \) and the algorithm stops. \( \square \)

We now prove that Algorithm D.1 computes the size of the new-shadow and the size of the new-shade of the \( p \)th set in squashed order.

Lemma D.7. Let \( B \) be the \( p \)th \( k \)-set in squashed order, \( k \in \mathbb{Z}^+ \), \( 1 \leq p \leq \binom{n}{k} \). Then, in Algorithm D.1, \( I(p) = |\Delta NB| \).

Proof. Let \( B \) be as in the statement of the lemma. For \( p = 1 \), \( I(1) = k = |\Delta NF_k(1)| \) by Corollary 2.41. Hence assume that \( p > 1 \).

Let \( \sum_{i=1}^{k} \binom{n}{i} \) be the \( k \)-binomial representation of \( p - 1 \). By Lemmas D.3 and D.4, \( I(p) + 1 \) is the smallest \( i \) for which \( \sigma(p - 1)[i] > 0 \). This implies that \( I(p) + 1 = t \) by Lemma D.5. Thus \( |\Delta NB| = t - 1 = I(p) \) by Lemma 2.62.(i). This proves Lemma D.7. \( \square \)

Lemma D.8. Let \( B \) be the \( p \)th \( k \)-set in squashed order, \( k \in \mathbb{Z}^+ \), \( 1 \leq p \leq \binom{n}{k} \). Then, in Algorithm D.1, \( Z(p) = |\nabla NB| \).

Proof. Let \( B \) be as in the statement of the lemma. For \( p = 1 \), \( Z(1) = 0 = |\nabla NF_k(1)| \) by Lemma 2.60.(ii) as \( 1 \in B \). Hence assume that \( p > 1 \).

Let \( \sum_{i=1}^{k} \binom{n}{i} \) be the \( k \)-binomial representation of \( p - 1 \). By Lemma D.5 \( Z(p) = \sigma(p - 1)[1] = a_t - 1 + 1 = a_t \) if \( t = 1 \) or \( 0 \) if \( t > 1 \). By Lemma 2.62.(ii) \( |\nabla NB| = a_t \) if \( t = 1 \) or \( |\nabla NB| = 0 \) if \( t > 1 \). Thus \( Z(p) = |\nabla NB| \) as required. \( \square \)

See Table D.1 for an example run of Algorithm D.1.
Appendix D. The Algorithm Used in the Proof of the 3-Levels Result

D.2 Implementation

The program shown on page 217 is an implementation of Algorithm D.1. Its output is given on page 221. This is the implementation which is used to prove Proposition 7.14 which states that Theorem 7.1-(i) holds for values of \( n \leq 32 \). We briefly describe the program by examining its main functions. For given \( n \) and \( k \), it computes the function \( \frac{\|L_{n,k+1}(p)\| + \|\triangle_N L_{n,k}(p)\|}{p} \) for each value of \( p \) between 1 and \( \binom{n}{k} \). The range of \( k \) is from \( \left\lfloor \frac{n+1}{2} \right\rfloor \) to \( n-2 \), and \( n \) ranges from \( N \) to \( M \) where \( N \) and \( M \) are any integers.

The range of \( k \) chosen is appropriate by Proposition 7.13 and the following lemma.

**Lemma D.9.** Theorem 7.1 holds for \( k = n - 1 \).

**Proof.** When \( k = n - 1 \) then \( p = 1 \) and, by Observations 2.9 and 2.7 and Corollary 2.41, \( |\triangle_N L_{n,k+1}(p)| + |\nabla L_{n,k-1}(p)| = |\triangle_{F_{n,n}}(1)| + |\nabla L_{n,n-1}(1)| = n + 2 > 2p \) for \( n \geq 2 \). Thus Theorem 7.1 holds for \( k = n - 1 \). \( \square \)

**The Function next_term.shade()**

The function `next_term.shade()` outputs the size of the new-shade of the \( p \)-th \( (k-1) \)-set in antilexicographic order. The function is a partial implementation of Algorithm D.1: Step 1 of Algorithm D.1 is performed in `main()`, while \( Z(p) \) is not implemented as it serves no purpose here. Note that \( p \) is implemented by `nbr_sets`.

I implements \( I(p-1) \) and \( I(p) \). `shade` is a \( (n-k+2) \)-element array whose first element `shade[0]` is left unused. `shade` is an implementation of \( \sigma(p) \) and stores the \( (n-k+1) \)-binomial representation of `nbr_sets` = \( p \) by Lemma D.5. `z` implements \( I(p) \). Let \( B \) be the \( p \)-th \( (n-k+1) \)-set in squashed order. Then \( z = |\triangle_N B| \) by Lemma D.7. By Lemma 2.21.(i) and Observation 2.4, it follows that \( z = |\nabla_N B'| \) where \( B' \) is the \( p \)-th \( (k-1) \)-set in antilexicographic order.
The Function \texttt{next\_term\_shadow()} 

The function \texttt{next\_term\_shadow()} outputs the size of the new-shadow of the $p$th $(k+1)$-set in antilexicographic order. The function is another partial implementation of Algorithm D.1: Step 1 is performed in \texttt{main()}.

\texttt{J} implements $I(p-1)$ and $I(p)$. \texttt{shadow} is a $(n-k)$-element array whose first element \texttt{shadow[0]} is left unused. \texttt{shadow} is an implementation of $\sigma(p)$ and stores the $(n-k-1)$-binomial representation of $p$ by Lemma D.5. Note that $n-k-1 > 0$ as $k < n-1$. \texttt{z} implements $Z(p)$ and, by Lemma D.8, $z = |\nabla_N B|$ where $B$ is the $p$th $(n-k-1)$-set in squashed order. Therefore $z = |\Delta_N B|$ where $B$ is the $p$th $(k+1)$-set in antilexicographic order by Lemma 2.21.(i) and Observation 2.4.

The Function \texttt{main()} 

It is now easy to see that $\text{sum} = |\Delta_N L_{n,k+1}(p)| + |\nabla_N L_{n,k-1}(p)|$ and that $\text{min\_func} = \min_{1 \leq p \leq \min\{\binom{n}{k+1},\binom{n}{k-1}\}} \frac{|\nabla_N L_{k-1}(p)|}{p} + \frac{|\Delta_N L_{k+1}(p)|}{p}$. The output of the program (see Page 221) for the values of $n \leq 32$ shows that $\text{min\_func} > 2$ for $k, \left\lceil \frac{n+1}{3} \right\rceil \leq k < n-1$. Therefore, $|\nabla_N L_{k-1}(p)| + |\Delta_N L_{k+1}(p)| > 2p$ for $n \leq 32$, thus proving Proposition 7.14.

Output of the Program 

For each $n$ and $k$, $n \leq 32$ and $\left\lceil \frac{n+1}{3} \right\rceil \leq k < n-1$, the output on Page 221 consists of (1) $\text{min}$ which is $\text{min\_func}$ as described in the previous paragraph, (2) $\text{nbr\_sets}$ which is the number of sets for which $\text{min}$ is output, (3) $\binom{n}{k+1} = \binom{n}{k+1}$ and (4) $\binom{n}{k-1} = \binom{n}{k-1}$.

Complexity Issues & Program Execution Environment 

Although Algorithm D.1 can be regarded as ‘efficient’ (see An Alternative Algorithm below) the complexity of the program as presented Page 217 is $O(2^n)$. The program was compiled with the GNU CC 2.7.2 compiler and was run on a DIGITAL Alphaserver 2000 4/333. It took 102 hours (4 days 6 hours) with an average of 90%
CPU usage to execute the program for the values of $n$ ranging from 2 to 32. It is worth noting that the case $n = 31$ took 22 hours to compute, and the case $n = 32$ took 34 hours. The largest $n$ for which the program was run was $n = 36$. On the same machine, with an average of 80% CPU usage, this run took approximately 6 weeks to execute!

### The Fidelity Problem

The program was run on several different machines apart from the Alphaserver mentioned above. For all runs of the program there was agreement for the value of `nbr_sets` and consequently of $\min$, since the value of `nbr_sets` is dependent on the value of $\min = \min_{func}$. The value of $\min_{func}$ partly depends on the algorithms used in floating-point arithmetic and on the underlying floating-point representation. It may therefore be the case that a different output to the one given here can be observed if the program is compiled and run in a different environment than the one described above.

### Further Comments

Examining the output on Page 221 one observes that the inequality in Theorem 7.1.(i) is tightest for $k = \frac{n}{2} - 1$ or $k = \frac{n}{2}$ when $n$ is even and for $k = \frac{n-1}{2}$ when $n$ is odd. In general, the inequality is tightest for values of $k$ close to $\frac{n}{2}$, and for values of $p$ very close to $\binom{n}{k-1}$ for $k < \frac{n}{2}$. Moreover, the inequality gets tighter for these values of $k$ as $n$ increases.

These observations are supported by the proof of Theorem 7.1, especially by the intricate proof of Proposition 7.22 which shows that Theorem 7.1.(i) holds for those values of $p$ closest to $\binom{n}{k-1}$ for $k \leq \frac{n}{2}$ and $k$ close to $\frac{n}{2}$. Careful examination of the proof of Proposition 7.22 shows that, for a substantial number, $p'$ say, of the sets under consideration, it is actually possible that $\frac{\| \nabla S_{k_{n-1}}[p']\|_H^{\nabla S_{k_{n-1}}[p']} L_{k_{n-1}}[p']}{\| L_{k_{n-1}}[p'] \|} = 2$ (see Lemma 7.29 which is used to show that the pair $(\mathcal{S}_1, \mathcal{S}_2)$ has property P in the proof of Lemma 7.32). It is not difficult to show that this is precisely the case when $p$ is close to $\binom{n}{k-1}$ and when $k = \frac{n}{2} - 1$ for $n$ even and $k = \frac{n-1}{2}$ for $n$ odd, the apparently
Appendix D. The Algorithm Used in the Proof of the 3-Lemma Result

‘worst’ cases when \(n\) is large, that is, the cases for which the function \(\text{min} \_\text{func}\) seems to attain its minimum. It seems reasonably easy to prove that Theorem 7.1 holds in the two cases just mentioned; they would require a much shorter proof than that of Proposition 7.22. Incidentally, it may be interesting to try to prove that the function \(\text{min} \_\text{func}\) attains its minimum for those values of \(n\) and \(k\) (with the added condition that \(n\) be sufficiently large), and then try to prove Theorem 7.1 from there on. We were however unable to make any progress in that direction.

Let \(f(n, k)\) be the value of \(\text{min} \_\text{func}\) for given \(n\) and \(k\). In the listing observe that \(f(n, k + 1) > f(n, k)\) for \(k \geq \frac{n}{2}\) when \(n\) is even and for \(k \geq \frac{n-1}{2}\) when \(n\) is odd. Observe also that \(f(n + 1, k + 1) = f(n, k)\) for \(k \geq \frac{n}{2}, n\) even, and for \(k \geq \frac{n-1}{2}, n\) odd. As \(\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}\), this makes intuitive sense if one assumes that \(f(n + 1, k + 1) = \min \{f(n, k + 1), f(n, k)\}\).

An Alternative Algorithm

We wrote another algorithm to compute the function \(\left[\sum_{L_k=1}^{L_{k+1}}(p)\right] + \left[\sum_{L_{k+1}}^{L_{k+2}}(p)\right]\). The details of the algorithm are not included here. It suffices to say that this algorithm computes \(\left|\nabla L_{k+1}(p)\right| = \left|\nabla L_{k+1}(p)\right| = \left|\Delta F_{n-k+1}(p)\right|\) by finding the \((n-k+1)\)-binomial representation of \(p\) and applying Theorem 2.39. Further, the algorithm computes \(\left|\Delta L_{k+1}(p)\right|\) by using the fact that \(\left|\Delta L_{k+1}(p)\right| = \binom{n}{k} - \left|\Delta F_{k+1}(\binom{n}{k} - p)\right|\) and by finding the \((k+1)\)-binomial representation of \((\binom{n}{k} - p)\). Therefore, for each \(p\), new \((n-k+1)\) and \((k+1)\)-binomial representations have to be computed. It can be easily seen that this algorithm is less efficient than Algorithm D.1. The algorithm just described is ‘memoryless’ while Algorithm D.1 uses known results about \(p - 1\) \((I(p - 1), \sigma(p - 1)\) and \(Z(p - 1))\) to derive new results \((I(p), \sigma(p)\) and \(Z(p))\) about \(p\). It must however be said that an implementation of this ‘inefficient’ algorithm was run for values of \(n \leq 26\) and that the output coincides with the output given here for the function \(\left[\sum_{L_k=1}^{L_{k+1}}(p)\right] + \left[\sum_{L_{k+1}}^{L_{k+2}}(p)\right]\) for these values of \(n\).
Implementation of Algorithm D.1

/*
 * Author: Paulette Lieby Copyright (C) June 1996
 * test if 
 * shadow_N L_{n,k+1} p + shade_N L_{n,k-1} p > 2p
 */

#include <iostream.h>
#include <stdlib.h>
#include <iomanip.h>
#include <Integer.h>
#include <limits.h>

int next_term_shade(); // computes shade_N L_{n,k-1} pth set
int next_term_shadow(); // computes shadow_N L_{n,k+1} pth set

void init(); // initialise arrays
void print_bounds(int, int); // print (n choose k+1) and (n choose k-1)
Integer fact(long int); // factorial function
Integer C(long int, long int); // binomial coefficient

int I, J, n, k; // need to be global, see next_term_shade() and shadow()
int *shade, *shadow; // arrays:
    // shade: (n-k+1)-binomial representation of nbr_sets
    // shadow: (n-k-1)-binomial representation of nbr_sets

main(long int argc, char* argv[]) {  
    long int N, M;
    Integer sum, nbr_sets; // sum accumulates
    Integer min_nbr_sets; // shadow_N L_{n,k+1} (nbr_sets)
    double min_func, r; // + shade_N L_{n,k-1} (nbr_sets)
    int sde, sdo;

    if (argc == 3) {
        N = atoi(argv[1]);
        M = atoi(argv[2]);
        cout << "\n";
    }
    else {
        cout << "entry format : <run> <from> <to> \n";
    }
for (n = N; n <= M; n++) {
    cout << " n = " << setw(5) << n << "\n";
    for (k = (n+1)/3; k < n-1; k++) {
        min_func = INT_MAX; // begin initialisation
        sum = nbr_sets = min_nbr_sets = 0;
        I = n-k+1; // I and J will
        J = n-k-1; // index *shade and *shadow
                    // respectively
        shade = new int[n-k+1+1];
        shadow = new int[n-k-1+1];
        init(); // end initialisation
        while
            ((sde=next_term_shade()) != -1
              && (sdo=next_term_shadow()) != -1) {
                sum = sum + sde + sdo;
                nbr_sets = nbr_sets + 1;
                r = ratio(sum,nbr_sets);
                if (r < min_func) {
                    min_func = r;
                    min_nbr_sets = nbr_sets;
                }
            }
        cout << " k = " << setw(5) << k; // output bit
        cout << " min = ";
        cout << setw(8) << setprecision(4) << min_func;
        cout << " for nbr_sets ";
        cout << setw(12) << min_nbr_sets;
        delete [] shade; delete [] shadow;
        print_bounds(n,k);
        cout << "\n";
    }
    cout << "\n";
}
} // end main()
Appendix D. The Algorithm Used in the Proof of the 3-Levels Result

```c
/* computes new-shade */
int next_term_shade()
{
    int j, z;

    if (shade[n-k+1] == k) {
        return -1;  // end
    }

    z = I;

    if (I == 0) {
        for (j=1; j < n-k+1 && shade[j] == shade[j+1]; j++)
        {
        }
        I = j;
        for (j=1; j < I; j++) {
            shade[j] = 0;
        }
    }
    else {}  // (I > 0)
    shade[I]++;
    I--;  // prepare for next term
    return z;
}  // end next_term_shade()

/* computes new-shadow */
int next_term_shadow()
{
    int j, z;

    if (shadow[n-k-2] == k+2) {
        return -1;  // end
    }

    z = shadow[1];

    if (J == 0) {
        for (j=1; j < n-k-1 && shadow[j] == shadow[j+1]; j++)
        {
        }
        J = j;
        for (j=1; j < J; j++) {
            shadow[j] = 0;
        }
    }
    else {}  // (J > 0)
    shadow[J]++;
    J--;  // prepare for next term
    return z;
}  // end next_term_shadow()
```
/* initialise the arrays */
void init()
{
    int i;
    for (i = 0; i < n-k+2; i++) {
        shade[i] = 0;
    }
    for (i = 0; i < n-k; i++) {
        shadow[i] = 0;
    }
}

/*/ print (n choose k+1) and (n choose k-1) */
void print_bounds(int n, int k)
{
    Integer px1, px2, px3, px4;

    px1 = (C(n,k+1));
    px2 = (C(n,k-1));

    cout << " (n k+1) " << setx(12) << px1;
    cout << " (n k-1) " << setx(12) << px2;
}

/*/ factorial function */
Integer fact(long int n)
{
    if (n)
    
    return n * fact(n-1);
    return 1;
}

/*/ binomial coefficient */
Integer C(long int n, long int k)
{
    Integer CC = 1;
    if (k < 0)
    
    return 0;
    if (n == k || k == 0)
    
    return CC; /* speed up */
else {
    for (long int i = n; i >= n - k + 1; i--)
        CC *= i;
    CC /= fact(k);
    return CC;
}
Output for $n \leq 32$

$$\begin{align*}
n &= 2 \\
n &= 3 \\
k &= 1 \text{ min } = 3 \text{ for nbr_sets } 1 \quad (n \ k+1) \ 3 \quad (n \ k-1) \ 1 \\
n &= 4 \\
k &= 1 \text{ min } = 4 \text{ for nbr_sets } 1 \quad (n \ k+1) \ 6 \quad (n \ k-1) \ 1 \\
k &= 2 \text{ min } = 3 \text{ for nbr_sets } 1 \quad (n \ k+1) \ 4 \quad (n \ k-1) \ 4 \\
n &= 5 \\
k &= 2 \text{ min } = 2.4 \text{ for nbr_sets } 5 \quad (n \ k+1) \ 10 \quad (n \ k-1) \ 5 \\
k &= 3 \text{ min } = 3 \text{ for nbr_sets } 1 \quad (n \ k+1) \ 5 \quad (n \ k-1) \ 10 \\
n &= 6 \\
k &= 2 \text{ min } = 2.667 \text{ for nbr_sets } 6 \quad (n \ k+1) \ 20 \quad (n \ k-1) \ 6 \\
k &= 3 \text{ min } = 2.4 \text{ for nbr_sets } 5 \quad (n \ k+1) \ 15 \quad (n \ k-1) \ 15 \\
k &= 4 \text{ min } = 3 \text{ for nbr_sets } 1 \quad (n \ k+1) \ 6 \quad (n \ k-1) \ 20 \\
n &= 7 \\
k &= 2 \text{ min } = 3.143 \text{ for nbr_sets } 7 \quad (n \ k+1) \ 35 \quad (n \ k-1) \ 7 \\
k &= 3 \text{ min } = 3.381 \text{ for nbr_sets } 24 \quad (n \ k+1) \ 35 \quad (n \ k-1) \ 21 \\
k &= 4 \text{ min } = 2.4 \text{ for nbr_sets } 5 \quad (n \ k+1) \ 21 \quad (n \ k-1) \ 35 \\
k &= 5 \text{ min } = 3 \text{ for nbr_sets } 1 \quad (n \ k+1) \ 7 \quad (n \ k-1) \ 35 \\
n &= 8 \\
k &= 3 \text{ min } = 2.429 \text{ for nbr_sets } 28 \quad (n \ k+1) \ 70 \quad (n \ k-1) \ 28 \\
k &= 4 \text{ min } = 3.381 \text{ for nbr_sets } 24 \quad (n \ k+1) \ 56 \quad (n \ k-1) \ 56 \\
k &= 5 \text{ min } = 2.4 \text{ for nbr_sets } 5 \quad (n \ k+1) \ 28 \quad (n \ k-1) \ 70 \\
k &= 6 \text{ min } = 3 \text{ for_nbr_sets } 1 \quad (n \ k+1) \ 8 \quad (n \ k-1) \ 56 \\
n &= 9 \\
k &= 3 \text{ min } = 2.694 \text{ for_nbr_sets } 36 \quad (n \ k+1) \ 126 \quad (n \ k-1) \ 36 \\
k &= 4 \text{ min } = 2.238 \text{ for_nbr_sets } 84 \quad (n \ k+1) \ 126 \quad (n \ k-1) \ 84 \\
k &= 5 \text{ min } = 3.381 \text{ for_nbr_sets } 24 \quad (n \ k+1) \ 84 \quad (n \ k-1) \ 126 \\
k &= 6 \text{ min } = 2.4 \text{ for_nbr_sets } 5 \quad (n \ k+1) \ 36 \quad (n \ k-1) \ 126 \\
k &= 7 \text{ min } = 3 \text{ for_nbr_sets } 1 \quad (n \ k+1) \ 9 \quad (n \ k-1) \ 84 \\
n &= 10 \\
k &= 3 \text{ min } = 2.911 \text{ for_nbr_sets } 45 \quad (n \ k+1) \ 210 \quad (n \ k-1) \ 45 \\
k &= 4 \text{ min } = 2.342 \text{ for_nbr_sets } 120 \quad (n \ k+1) \ 210 \quad (n \ k-1) \ 120 \\
k &= 5 \text{ min } = 2.238 \text{ for_nbr_sets } 84 \quad (n \ k+1) \ 210 \quad (n \ k-1) \ 210 \\
k &= 6 \text{ min } = 3.381 \text{ for_nbr_sets } 24 \quad (n \ k+1) \ 120 \quad (n \ k-1) \ 210 \\
k &= 7 \text{ min } = 2.4 \text{ for_nbr_sets } 5 \quad (n \ k+1) \ 45 \quad (n \ k-1) \ 210 \\
k &= 8 \text{ min } = 3 \text{ for_nbr_sets } 1 \quad (n \ k+1) \ 10 \quad (n \ k-1) \ 120 \\
n &= 11 \\
k &= 4 \text{ min } = 2.455 \text{ fornbr_sets } 165 \quad (n \ k+1) \ 462 \quad (n \ k-1) \ 165 \\
k &= 5 \text{ min } = 2.212 \text{ for_nbr_sets } 330 \quad (n \ k+1) \ 462 \quad (n \ k-1) \ 330 \\
k &= 6 \text{ min } = 2.238 \text{ for_nbr_sets } 84 \quad (n \ k+1) \ 330 \quad (n \ k-1) \ 462 \\
k &= 7 \text{ min } = 3.381 \text{ for_nbr_sets } 24 \quad (n \ k+1) \ 165 \quad (n \ k-1) \ 462 \\
k &= 8 \text{ min } = 2.4 \text{ for_nbr_sets } 5 \quad (n \ k+1) \ 55 \quad (n \ k-1) \ 330 \\
\end{align*}$$
### Appendix D: The Algorithm Used in the Proof of the 3-Levels Result

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<tr>
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### Appendix D. The Algorithm Used in the Proof of the 3-Levels Result

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\(n = 18\)

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\(n = 19\)

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### Appendix D. The Algorithm Used in the Proof of the 3-Leves Result

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### Appendix D. The Algorithm Used in the Proof of the 3-Lefschetz Result

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### Appendix D. The Algorithm Used in the Proof of the 3-Levels Result

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Appendix E

More Lemmas for the Proof of the 3-Levels Result
This appendix consists of the lemmas involving algebraic inequalities which are used to prove Theorem 7.1. Below is a table of correspondence between the lemmas and the proposition or lemma in which they are referred to.

<table>
<thead>
<tr>
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<th>Proposition 7.19</th>
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<td>Lemma E.2</td>
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All of the following lemmas, with the exception of Lemma E.3, are stated for values of \( n > 32 \). However, in most cases, the lemmas hold for smaller values of \( n \). We acknowledge the help of J. Simpson and I. Roberts in writing some of the proofs.

**Lemma E.1.** Let \( n \) and \( k \in \mathbb{Z}^+ \) be such that \( n > 32 \) and \( \frac{n^2-3}{3} < k \leq \frac{n}{2} \). Then

\[
\left( \frac{n-1}{k-2} \right) < \left( \frac{n-2}{k-1} \right)
\]

implies that

\[
\frac{k-1}{n-k} + \frac{n-k-1}{k} \geq 2.
\]  

*(E.1)*

**Proof.** We wish to prove that
\[ \frac{(n - 1)(k - 1)}{(n - k + 1)(n - k)} < 1 \] \hspace{1cm} (E.2)

implies (E.1). This is the same as showing that (E.2) fails when (E.1) fails.

(E.2) can be rewritten as \( 4k^2 - 4nk + n^2 - n \geq 0 \) and (E.1) can be rewritten \( k^2 - 3nk + n^2 + 2n - 1 > 0 \). Therefore (E.1) fails for values of \( k \) in the range \( \frac{n - \sqrt{n}}{2} < k \leq \frac{n}{2} \) and (E.2) fails for values of \( k \) in the range \( \frac{3n - \sqrt{5n^2 - 8n + 4}}{2} \leq k \leq \frac{n}{2} \). For values of \( n > 32 \) we have that \( n - \sqrt{n} \geq 3n - \sqrt{5n^2 - 8n + 4} \), thus Lemma E.1 holds.  

\[ \text{Lemma E.2. Let } n \text{ and } k \in \mathbb{Z}^+ \text{ be such that } n > 32 \text{ and } \frac{n + 1}{3} < k \leq \frac{n}{2}. \text{ Then} \]
\[ \left( \frac{n - 1}{k - 2} \right) < \left( \frac{n - 2}{k - 1} \right) \]

implies that
\[ \left( \frac{n - 2}{k} \right) > \left( \frac{n - 2}{k - 1} \right). \]

\[ \begin{align*}
\text{Proof. Lemma E.2 can be rewritten as} \hspace{1cm} \\
\frac{(n - 1)(k - 1)}{(n - k + 1)(n - k)} < 1 \hspace{1cm} (E.3) \\
\text{implies that} \hspace{1cm} \\
\frac{n - k - 1}{k} > 1. \hspace{1cm} (E.4)
\end{align*} \]

(E.3) can be rewritten as \( k^2 - 3nk + n^2 + 2n - 1 > 0 \) and this holds for values of \( k \) such that \( k < \frac{3n - \sqrt{5n^2 - 8n + 4}}{2} \) and \( k > \frac{3n + \sqrt{5n^2 - 8n + 4}}{2} > n \). As \( k < n \), we only need to consider the bound \( k < \frac{3n - \sqrt{5n^2 - 8n + 4}}{2} \). Assume that \( k < \frac{3n - \sqrt{5n^2 - 8n + 4}}{2} \).

Note that \( 5n^2 - 8n + 4 > \left( \frac{13}{6} n \right)^2 \) for \( n > 25 \). Thus (E.3) implies that \( k < \frac{3n - \sqrt{5n^2 - 8n + 4}}{2} = \frac{5}{12} n \) for \( n > 25 \). In which case \( k < \frac{5}{12} n \) implies that \( k < \frac{4n - 1}{2} \), thus (E.4) holds.

The next lemma is not referred to in the proof of Theorem 7.1 but it is required to prove Lemmas E.4 and E.5 below.
Lemma E.3. Let $n, k \in \mathbb{Z}^+$ be such that $n > 53$, and $\frac{n+1}{3} < k \leq \frac{n}{2}$. Then
\[
\frac{(n-k+1)(n-k)(n-k-1)}{k(k-1)(k-2)} - \frac{n-k+1}{k-2} > 1
\]
implies that
\[
\frac{k-1}{n-k} + \frac{n-k-1}{k} \geq 2.
\]

Proof. For $n$ and $k$ fixed, by replacing $n-k$ by $m$, $\frac{n}{2} \leq m < \frac{2n-1}{3}$, \eqref{E.5} and \eqref{E.6} respectively become
\[
\frac{(m+1)m(m-1)}{k(k-1)(k-2)} - \frac{m+1}{k-2} > 1
\]
\[
\equiv \frac{(m+1)[m(m-1) - k(k-1)]}{k(k-1)(k-2)} > 1
\]
and
\[
\frac{k-1}{m} + \frac{m-1}{k} \geq 2.
\]
\eqref{E.8} fails for $\frac{2m+1-\sqrt{8m+1}}{2} < k < \frac{2m+1+\sqrt{8m+1}}{2}$.

As $m + k = n$, the left-hand side of \eqref{E.7} is a decreasing function of $k$ for $n$ fixed. Also, if $k = K \geq \frac{2m+1-\sqrt{8m+1}}{2}$, then this latter inequality holds for the values of $k > K$.

Replacing $k$ by $\frac{2m+1-\sqrt{8m+1}}{2}$ in \eqref{E.7} gives
\[
2m^2 + 15m + 1 - (5m+1)\sqrt{8m+1} < 0.
\]
This fails for $m = 35$. Actually, it fails for $m \geq 35$ as the derivative of the left-hand side of \eqref{E.7}, $4m + 15 - 5\sqrt{8m+1} - \frac{4(5m+1)}{\sqrt{8m+1}}$ is positive for $m \geq 21$.

This proves Lemma E.3 for $m \geq 35$. That is, Lemma E.3 holds for $n = m + k \geq 35 + k$ for some $k$, $\frac{n+1}{3} < k \leq \frac{n}{2}$. For Lemma E.3 to hold for all values of $n = m + k$ with $\frac{n+1}{3} < k \leq \frac{n}{2}$ it is sufficient that $n = m + \frac{n}{2}$ and $m \geq 35$, that is, it is sufficient that $n \geq 70$. A computer check shows that it holds for $53 < n < 70$. \qed
Lemma E.4. Let \( n, k \in \mathbb{Z}^+ \) be such that \( n > 32 \), and \( \frac{n+1}{3} < k \leq \frac{n}{2} \). Then
\[
\frac{k-1}{n-k} + \frac{n-k-1}{k} < 2
\]
and
\[
\frac{n-2}{k} - \frac{n-2}{k-2} > \frac{n-2}{k-3}
\]
imply that
\[
\left( \frac{n-1}{k} \right) \geq \left( \frac{n-1}{k-1} \right) + \left( \frac{n-3}{k-2} \right).
\]

Proof. Lemma E.4 can be rewritten as
\[
\frac{k-1}{n-k} + \frac{n-k-1}{k} < 2 \quad (E.10)
\]
and
\[
\frac{(n-k+1)(n-k)(n-k-1)}{k(k-1)(k-2)} - \frac{n-k+1}{k-2} > 1 \quad (E.11)
\]
imply that
\[
\frac{k}{n-k} + \frac{k(k-1)}{(n-1)(n-2)} \leq 1. \quad (E.12)
\]
By Lemma E.3, the truth of (E.11) implies that (E.10) is false for \( n > 53 \), in which case there is nothing to prove. A computer check shows that Lemma E.4 holds for the values of \( n \) such that \( 18 < n \leq 53 \). \( \square \)

Lemma E.5. Let \( n, k \in \mathbb{Z}^+ \) be such that \( n > 32 \), and \( \frac{n+1}{3} < k \leq \frac{n}{2} \). Then
\[
\frac{k-1}{n-k} + \frac{n-k-1}{k} < 2
\]
and
\[
\frac{n-2}{k} - \frac{n-2}{k-2} > \frac{n-2}{k-3}
\]
imply that
\[
\left( \frac{n-2}{k} \right) - \left( \frac{n-2}{k-2} \right) - \left( \frac{n-3}{k-2} \right) \leq \left( \frac{n-2}{k-3} \right).
\]
Appendix E. More Lemmas for the Proof of the 3-L evels Result

Proof. As in Lemma E.4, the hypothesis is only true for \( n \leq 53 \) by Lemma E.3. Thus Lemma E.5 holds for \( n > 53 \). Exhaustive computations show that it holds for \( 27 < n \leq 53 \). \[\square\]

Lemma E.6. Let \( n, k \in \mathbb{Z}^+, \, i \in \mathbb{N} \), be such that \( n > 32, \) \( \frac{n+1}{3} < k \leq \frac{n}{2} \), and \( 0 \leq i < k - 4 \). Then

\[
\left( \frac{n-4-i}{k-3-i} \right) > \left( \frac{n-3-i}{k-4-i} \right)
\]

implies that

\[
\left( \frac{n-5-i}{k-5-i} \right) + \left( \frac{n-5-i}{k-3-i} \right) \geq 2 \left( \frac{n-5-i}{k-4-i} \right).
\]

Proof. Lemma E.6 can be rewritten as

\[
\frac{(n-k+1)(n-k)}{(n-3-i)(k-3-i)} > 1 \quad \text{(E.13)}
\]

implies that

\[
\frac{n-k-1}{k-3-i} + \frac{k-4-i}{n-k} \geq 2. \quad \text{(E.14)}
\]

For \( n, k \) and \( i \) fixed, perform a change of variables using \( j = k-3-i \) and \( m = n-k \), \( 1 < j \leq k-3, \) \( \frac{n}{2} \leq m < \frac{2n-1}{3} \). (E.13) and (E.14) respectively become:

\[
\frac{(m+1)m}{(m+j)j} > 1 \quad \text{(E.15)}
\]

and

\[
\frac{m-1}{j} + \frac{j-1}{m} \geq 2. \quad \text{(E.16)}
\]

Lemma E.6 holds if the falsity of (E.16) implies the falsity of (E.15). (E.16) is false for the values of \( j \) such that \( \frac{2m+1-\sqrt{8m+1}}{2} < j < \frac{2m+1+\sqrt{8m+1}}{2} \).

Note that \( m \) and \( j \) are dependent variables as \( m+j = n-3-i \). As \( n \) and \( i \) are fixed, the left-hand side of (E.15) is decreasing when \( j \) is increasing. Moreover it is easy to show that if \( j = J \geq \frac{2m+1-\sqrt{8m+1}}{2} \) then this last inequality holds also for the values of \( j > J \).
Appendix E. More Lemmas for the Proof of the 3-Levels Result

We show that for \( j = \frac{2m+1 - \sqrt{8m+1}}{2} \), (E.15) is false as well, implying that (E.15) is false for all the values of \( j \) for which (E.16) is false. Replacing \( j \) by \( \frac{2m+1 - \sqrt{8m+1}}{2} \) in (E.15) we obtain

\[
4m^2 + 10m + 2 - (6m + 2)\sqrt{8m + 1} < 0. \tag{E.17}
\]

(E.17) fails for \( m = 14 \). In fact (E.17) fails for values of \( m \geq 14 \) as its left-hand side is an increasing function with respect to \( m \). To see this, consider its derivative

\[
8m + 10 - 6\sqrt{8m + 1} - \frac{4(6m + 2)}{\sqrt{8m + 1}} \frac{1}{\sqrt{8m + 1}}
\]

which is positive for values of \( m \geq 10 \).

This proves Lemma E.6 for \( m \geq 14 \). By the same argument as the one used in the proof of Lemma E.3 we see that Lemma E.6 holds for \( n \geq 28 \). A computer check shows that it holds for \( 22 < n < 28 \).

\[\square\]

Lemma E.7. Let \( n, k \in \mathbb{Z}^+ \), \( i \in \mathbb{N} \), be such that \( n > 32, \frac{n+1}{3} < k \leq \frac{n}{2} \), and \( 0 \leq i < k - 4 \). Then

\[
\binom{n - 5 - i}{k - 3 - i} > \binom{n - 3 - i}{k - 4 - i}
\]

implies that

\[
\frac{n - k - 3}{k - 2 - i} > \frac{n - k}{k} \geq 0. \tag{E.18}
\]

Proof. Lemma E.7 can be rewritten as

\[
\frac{(n - k + 1)(n - k)(n - k - 1)}{(n - 3 - i)(n - 4 - i)(k - 3 - i)} > 1 \tag{E.19}
\]

implies (E.18).

It is easy to see that this holds for values of \( i \geq 1 \). It also holds for \( k \leq \frac{2n}{3} \) for \( i = 0 \).

Note that the left-hand side of (E.19) is decreasing when \( k \) is increasing.

(E.18) fails for \( k > \frac{2n}{3} \). Hence, proving that (E.19) fails for \( k = \frac{2n}{3} \) is enough to prove Lemma E.7. Replacing \( k \) by \( \frac{2n}{3} \) in (E.19) gives

\[
\frac{(3n + 5)(3n)(3n - 5)}{(5n - 15)(5n - 20)(2n - 15)} > 1
\]

\[
\equiv \quad 23n^5 - 725n^2 + 3300n - 4500 < 0. \tag{E.20}
\]
E. More Lemmas for the Proof of the 3-Levels Result

(E.20) fails for \( n = 27 \). Moreover, the derivative of the left-hand side of (E.20) is positive for \( n \geq 19 \). As a result, (E.20) fails for \( n \geq 27 \). This proves the lemma for \( n \geq 27 \). A computer check shows that it holds for \( 22 < n < 27 \).

Lemma E.8. Let \( n, k \in \mathbb{Z}^+ \), \( i \in \mathbb{N} \), be such that \( n > 32 \), \( \frac{n + 1}{3} < k \leq \frac{n}{2} \), and \( 0 \leq i < k - 4 \). Then

\[
\binom{n - 5 - i}{k - 3 - i} > \binom{n - 3 - i}{k - 4 - i}
\]

implies that

\[
\binom{n - 6 - i}{k - 5 - i} + \binom{n - 6 - i}{k - 3 - i} \geq 2 \binom{n - 6 - i}{k - 4 - i}.
\]

Proof. Lemma E.8 can be rewritten as:

\[
\frac{(n - k + 1)(n - k)(n - k - 1)}{(n - 3 - i)(n - 4 - i)(k - 3 - i)} > 1 \quad \text{(E.21)}
\]

implies that

\[
\frac{n - k - 2}{k - 3 - i} + \frac{k - 4 - i}{n - k - 1} \geq 2. \quad \text{(E.22)}
\]

For \( n, k \) and \( i \) fixed perform a change of variables using \( j = k - 3 - i \) and \( m = n - k \), \( 1 < j \leq k - 3 \), \( \frac{n}{2} \leq m < \frac{2n - 1}{3} \). (E.21) and (E.22) respectively become:

\[
\frac{(m + 1)m}{(m + j)j} \cdot \frac{(m - 1)}{m + j - 1} > 1 \quad \text{(E.23)}
\]

and

\[
\frac{m - 2}{j} + \frac{j - 1}{m - 1} \geq 2. \quad \text{(E.24)}
\]

Lemma E.8 holds if the falsity of (E.24) implies the falsity of (E.23). (E.24) is false for the values of \( j \) such that \( \frac{2m - 1 - \sqrt{8m - 7}}{2} < j < \frac{2m - 1 + \sqrt{8m - 7}}{2} \).

Note that \( m \) and \( j \) are dependent variables as \( m + j = n - 3 - i \). When \( j \) is increasing the left-hand side of (E.23) is decreasing, and it is easily verified that if \( j = J \geq \frac{2m - 1 - \sqrt{8m - 7}}{2} \), then this latter inequality also holds for values of \( j > J \).

We show that for \( j = \frac{2m - 1 - \sqrt{8m - 7}}{2} \), (E.23) is false as well, implying that (E.23) is false for all the values of \( j \) for which (E.24) is false, hence proving the lemma.
Appendix E. More Lemmas for the Proof of the 3-Levels Result

Note that, if
\[
\frac{(m+1)m}{(m+j)j} \leq 1
\]  
(E.25)
then (E.23) fails, as \(\frac{m+1}{m+j-1} < 1\) for \(j > 0\).

Replacing \(j\) by \(\frac{2m-1-\sqrt{8m-1}}{2}\) in (E.25) gives
\[
4m^2 - 2m - 6 - (6m - 2)\sqrt{8m - 7} \geq 0.
\]  
(E.26)

(E.26) holds for \(m = 18\). In fact (E.26) holds for values of \(m \geq 18\) as the left-hand side of (E.25) is an increasing function with respect to \(m\). To see this, consider its derivative \(8m - 2 - 6\sqrt{8m - 7} - \frac{4(6m - 2)}{\sqrt{8m - 7}}\), which is positive for values of \(m \geq 14\).

Hence (E.25) holds for \(m \geq 18\), that is, (E.23) fails for \(m \geq 18\). This proves the lemma for \(m \geq 18\). Replicating the argument used in the proof of Lemma E.3 shows that Lemma E.8 holds for \(n \geq 36\). A computer check shows that it holds for \(10 < n < 36\). \(\square\)

**Lemma E.9.** Let \(n, k \in \mathbb{Z}^+, i \in \mathbb{N}\), be such that \(n > 32\), \(\frac{n-1}{3} < k \leq \frac{n}{2}\), and \(0 \leq i < k - 4\). Then
\[
\left(\frac{n-6-i}{k-3-i}\right) > \left(\frac{n-3-i}{k-4-i}\right)
\]
implies that
\[
\left(\frac{n-6-i}{k-2-i}\right) \geq 2\left(\frac{n-6-i}{k-3-i}\right).
\]

**Proof.** Lemma E.9 is rewritten as
\[
\frac{(n-k+1)(n-k)(n-k-1)(n-k-2)}{(n-3-i)(n-4-i)(n-5-i)(k-3-i)} > 1
\]  
(E.27)
implies that
\[
\frac{n-k-3}{k-2-i} \geq 2.
\]  
(E.28)

For \(n, k\) and \(i\) fixed, perform a change of variables using \(j = k - 3 - i\) and \(m = n - k\), \(1 < j \leq k - 3\), \(\frac{m}{2} \leq m < \frac{2m-1}{3}\). (E.27) and (E.28) respectively become:
\[
\frac{(m+1)m(m-1)(m-2)}{(m+j)(m+j-1)(m+j-2)} > 1
\]  
(E.29)
Lemma E.9 holds if the falsity of (E.30) implies the falsity of (E.29). (E.30) is false for the values of $j$ such that $j > \frac{m-5}{2}$.

Note that $m+j = n-3-i$. As $n$ and $i$ are fixed, note that the left-hand side of (E.29) is decreasing when $j$ is increasing. Moreover, it is easily shown that if $j = J \geq \frac{m-5}{2}$ then this latter inequality also holds for values of $j > J$.

We show that for $j = \frac{m-5}{2}$ (E.29) is false as well, implying that (E.29) is false for all the values of $j$ for which (E.30) is false, hence proving the lemma. Replacing $j$ by $\frac{m-5}{2}$ in (E.29) gives

\[ 11m^4 - 292m^3 + 1390m^2 - 2492m + 1575 < 0. \]  

(E.31) fails for $m = 22$. In fact (E.29) fails for values of $m \geq 22$ as its left-hand side term is an increasing function with respect to $m$. The derivative of this term $44m^3 - 876m^2 + 2780m - 2492$ is positive for values of $m = 17$ and an increasing function of $m$, as its second derivative $132m^2 - 1752m + 2780$ is positive for $m \leq 1$ and $m \geq 12$.

This proves Lemma E.9 for $m \geq 22$. Again, by the same argument as the one used in the proof of Lemma E.3, it follows that Lemma E.9 holds for $n \geq 44$. A computer check shows that the lemma holds for $32 < n < 44$. \qed
References


REFERENCES


