Covering Separating Systems

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Thesis Declaration

I hereby declare that the work herein, now submitted as a thesis for the degree of Master of Science by research, is the result of my own investigations, and all references to ideas and work of the other researchers have been specifically acknowledged. I hereby certify that the work embodied in this thesis has not already been accepted in substance for any degree, and is not being currently submitted in candidature for any degree.

26.02.99

[Signature]

Oudone Phanalasy
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Abstract

This thesis considers two related structures on finite sets, namely separating systems and covering separating systems. These systems are considered from the point of view of combinatorial designs. Separating systems have been considered by previous authors but this is the first time that covering separating systems have been studied in detail.

A separating system on \([n] = \{1, \ldots, n\}\) is a collection of subsets of \([n]\), in which for each pair of elements of \([n]\), there exists a set in the collection which contains the first element but not the second or vice versa. A covering separating system on \([n]\) is a separating system on \([n]\) in which each element of \([n]\) occurs at least once. A covering separating system on \([n]\) in which each set has cardinality \(k\) is called a \((n, k)\) covering separating system. The size of a (covering) separating system is the number of sets in the collection. The volume of a (covering) separating system is the sum of the cardinalities of the sets in the collection. A minimal (covering) separating system on \([n]\) is one which contains the least possible number of sets.

The size of a minimal separating system has been determined for all \(n\) by Renyi. The size of minimal covering separating systems on \([n]\) and bounds or exact values on the size of a minimal \((n, k)\) covering separating system is determined here for all \(n\) and \(k\). Various results are derived on the volume of minimal separating systems and minimal covering separating systems on \([n]\), and on the volume of minimal \((n, k)\) covering separating systems.

A catalogue of all non-isomorphic minimal separating systems for \(n \leq 8\) and all non-isomorphic minimal covering separating systems for \(n \leq 7\) is given. Some conjectures concerning minimal \((n, k)\) covering separating systems are made.

The thesis concludes with a brief mention of the connection between covering separating systems and finite topologies. This provides one possible avenue for applications of the theory of covering separating systems.
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Chapter 1

Introduction
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1.1 Overview

There are several motivations for this thesis which concentrates on a study of covering separating systems. First, it is a natural extension of work of various authors on separating systems and completely separating systems. Second, some types of covering separating systems are precisely related to certain types of topological spaces. It is hoped that the consideration of covering separating systems in this thesis may be useful in the study of finite topologies. Third, there is a duality between some types of covering separating systems and certain collections of subsets on a related universal set. It is hoped that consideration of covering separating systems will aid in the development of the theory of the related collections.

A separating system (or \((n)SS\)) \(\mathcal{S}\) on a finite set \([n] = \{1, \ldots, n\}\) is a collection of subsets of \([n]\) in which for each pair of elements of \([n]\) there is a set in \(\mathcal{S}\) which contains the first element but not the second or vice versa. A covering separating system (or \((n)KSS\)) is a \((n)SS\) in which every element of \([n]\) occurs. A completely separating system (or \((n)CSS\)) on \([n]\) is a collection of subsets of \([n]\) in which for each pair of elements \(a, b \in [n]\) there is a set in \(\mathcal{C}\) which contains \(a\) but not \(b\) and vice versa.

Separating systems and completely separating systems have been considered by various authors. Covering separating systems are studied for the first time in this thesis. A major aim of this thesis is to determine the minimum size of \((n)KSSs\) with and without restrictions on the size of the sets in the \(KSSs\). The approaches used to address this problem were guided by the work of Ramsay et al. [16, 17] and Roberts [19, 20]. Results were enhanced by discussion with Roberts and Lieby.

Chapter 2 gives a brief review of some historical developments in the study of separating systems and completely separating systems and presents some new
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results on completely separating systems. Throughout the thesis, comparisons to corresponding results on CSSs will be made.

A \((n, h, k)KSS\) is a \((n)KSS\) in which each set has size at least \(h\) and at most \(k\). If \(h = k\) then a \((n, h, k)KSS\) is called a \((n, k)KSS\). The size of a minimal \((n)KSS\), \((n, h, k)KSS\), or \((n, k)KSS\) is denoted by \(K(n)\), \(K(n, h, k)\) or \(K(n, k)\) respectively.

In most cases in this thesis the derivation of results concerning minimal \((n)KSSs\) precede the derivation of results concerning \((n)SSs\). There are two reasons for this. Firstly the results on \((n)SSs\) are easily derived using results on \((n-1)KSSs\) and \((n)KSSs\). Secondly, if the reverse approach is taken, then it would be necessary to continually refer to minimal \((n)SSs\) in which each element occurs and to minimal \((n)SSs\) in which one element of \([n]\) is missing. This would add an unwanted duplication in the arguments used.

Some results on \((n)KSSs\) are derived in Chapter 3 and the dual of these collections is defined and considered briefly. Chapter 3 also gives the exact value for \(K(n)\) for all \(n\) and \(k\). Note that small results on minimal \((n, h, k)KSSs\) are occasionally included in this and later chapters.

The volume of a collection of sets is defined to be the sum of the cardinalities of the sets in the collection. Chapter 4 includes results on the minimum volume of minimal \((n)KSSs\). Chapter 5 derives results on the minimum volume of minimal \((n)SSs\) using the results in Chapter 4.

The results in Chapters 4 and 5 are useful in determining the number of non-isomorphic minimal \((n)KSSs\) for \(2 \leq n \leq 7\) and \((n)SSs\) for \(2 \leq n \leq 8\) and for some other values of \(n\). This is done in Chapter 6. Chapter 6 also derives a catalogue of non-isomorphic minimal \((n)KSSs\) which achieve \(K(n)\) for \(2 \leq n \leq 7\). It is explained how a catalogue of non-isomorphic minimal \((n)SSs\) for \(2 \leq n \leq 8\) can be made from the catalogue of non-isomorphic minimal \((n)KSSs\).
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A concise form of both catalogues appears in Appendix B. Although the sizes of minimal \((n)SSs\) have been known for some time, this is the first time that all non-isomorphic minimal \((n)SSs\) have been catalogued for \(n \leq 8\).

A study of minimum size \((n,k)KSSs\) is included in Chapter 7 and the value of \(K(n,k)\) is determined for all \(n\) with \(k \leq 4\) and \(n \geq \frac{k^2}{2}\). Chapter 8 and Appendix E extend the results in Chapter 7 to include the value of \(K(n,k)\) for all values of \(n\) for \(k \leq 9\) and for all values of \(n \leq 40\) with \(10 \leq k \leq 15\).

Chapter 9 presents two related conjectures concerning minimal \((n,k)CSSs\) and shows that the conjectures are valid for a number of values of \(n\) and \(k\). It also includes a table of values of \(K(n,k)\) for \(n \sim 40\) and \(k \sim 15\).

Chapter 10 briefly considers the relationship between \(KSSs\) and finite topologies and a brief motivation is given as to why this connection may be worth exploring at some future date.

Open problems and some ideas for future work are included throughout the thesis.

The thesis includes various appendices. A list of standard notation and terminology is given in Appendix A. Appendix B includes a catalogue of all non-isomorphic minimal \((n)KSSs\) for \(2 \leq n \leq 7\), and all non-isomorphic minimal \((n)SSs\) for \(2 \leq n \leq 8\). Appendix C contains example minimal \((n,k)KSSs\) which illustrate the construction used in Theorem 7.6. Appendix D contains example minimal \((n,k)KSSs\) which illustrate the construction used in Theorem 7.7. Appendix E is a continuation of the approach taken in Chapter 8. It determines the unknow values of \(K(n,k)\) for \(n \leq 40\) with \(10 \leq k \leq 15\). Appendices F and G present some example minimal \((n,k)KSSs\) described in Chapter 8.

Notation and definitions used in more than one chapter are included in this chapter. All sets considered in this thesis are finite. Where convenient the braces are left out when denoting sets. For example, the collection \{\{1,2,3\}, \{1,4\}\} may
be written as \{123, 14\}.

1.2 Definitions and Examples

Let \([n] = \{1, 2, \ldots, n\}\). Let \(C = \{A_1, A_2, \ldots, A_m\}\) be a collection of subsets of \([n]\). The volume of \(C\) is \(V(C) = \sum_{i=1}^{m} |A_i|\). The complement \(C'\) of \(C\) is \(C' = \{A \subseteq [n] : A' \in C\}\). A corresponding \((0,1)\)-array (or incidence array) of \(C\) is a \(m \times n\)-array \(M_{mn} = (m_{ij})\) (or simply \(M\)) defined by

\[
m_{ij} = \begin{cases} 
1 & \text{if } j \in A_i \\
0 & \text{if } j \notin A_i.
\end{cases}
\]

\(C\) is an antichain if \(A_i \not\subseteq A_j\) for each \(i, j \in [m]\). The dual of \(C\) is the collection \(C^* = \{X_1, X_2, \ldots, X_n\}\) of subsets of \([m]\) where \(X_j = \{t : j \in A_i\}\) for each \(j \in [n]\). Two collections \(C_1\) and \(C_2\) are said to be isomorphic, written \(C_1 \cong C_2\), if \(|C_1| = |C_2|\) and if there exists a permutation \(\pi\) on \([n]\) such that \(\pi(A) \in C_2\) for each \(A \in C_1\).

Let \(M_1, M_2, \ldots, M_q\) be arrays and assume that \(M_1\) has the largest number of rows of the \(M_i, i \leq i \leq q\). An augmented array of \(M_1, M_2, \ldots, M_q\) is an array \(M\), denoted by \([M_1, M_2, \ldots, M_q]\), which is obtained by writing side by side the arrays \(M_1, M_2, \ldots, M_q\) and adding spaces in the bottom rows if necessary, so that \(M\) has the same number of rows as \(M_1\).

An element \(a \in [n]\) is said to be separated from \(b \in [n]\) by \(C\) if there exists \(A \in C, a \in A, b \not\in A\). A separating system (or SS) \(C\) on \([n]\) is a collection of subsets of \([n]\) in which for any pair \((a, b) \in [n] \times [n]\), either \(a\) is separated from \(b\) or \(b\) is separated from \(a\). Such a collection is also called a \((n)\)SS. A covering separating system (or KSS) \(K\) on \([n]\) is \((n)\)SS in which for each \(a \in [n]\) there is an \(A \in K\) with \(a \in A\). Such a collection is also called a \((n)\)KSS. The elements of \([n]\) are said to be covering separated by \(K\). A completely
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A separating system (or CSS) \( C \) on \([n]\) is a collection of subsets of \([n]\) in which for any pair \((a, b) \in [n] \times [n]\), \(a\) is separated from \(b\) and \(b\) is separated from \(a\). Such a collection is also called a \((n)CSS\). The elements of \([n]\) are said to be completely separated by \(C\).

A SS, KSS or CSS \( C \) is said to be fair if there exists a positive integer \(p\) such that every element of \([n]\) occurs in either \(p\) or \(p+1\) sets of \(C\). If every element of \([n]\) occurs in exactly \(p\) sets of \(C\) then \(C\) is said to be equitable.

Observe that any CSS is a KSS and that any KSS is a SS, but not vice versa. For example, in \(C = \{\{1, 2\}, \{1, 3\}\}\), 1 is separated from 2 by the set \(\{1, 3\}\), but 2 is not separated from 1 so that \(C\) is a SS on \([3]\) and on \([4]\). \(C\) is a \((3)KSS\) but not a \((4)KSS\). \(C\) is not a CSS. Also observe that if \(C\) is a CSS or SS on \([n]\) then the complement \(C'\) is a CSS or SS respectively, but the complement of a KSS on \([n]\) is not always a KSS. For example, \(K = \{12, 13, 14\}\) is a KSS on \([4]\). The complement \(K' = \{34, 24, 23\}\) is not a KSS on \([4]\) since the element 1 is not in any member of \(K'\).

If the sets in a \((n)SS\), \((n)KSS\), or \((n)CSS\) have size at least \(h\) and at most \(k\), then the separating systems are denoted by \((n, h, k)SS\), \((n, h, k)KSS\), or \((n, h, k)CSS\) respectively. If all the sets in the separating systems have size \(k\), then the separating systems are denoted by \((n, k)SS\), \((n, k)KSS\), or \((n, k)CSS\) respectively.

Let \(S(n)\) be the collection of all SSs on \([n]\). Then \(S \in S(n)\) is said to be minimal if \(|S| = \min_{C \in S(n)} |C|\). Similar definitions of minimality apply to the other collections considered here.

The size of a minimal \((n)SS\), \((n)KSS\), or \((n)CSS\) is denoted by \(S(n)\), \(K(n)\), or \(R(n)\) respectively. Similarly, the sizes of minimal \((n, h, k)SSs\), \((n, h, k)KSSs\), and \((n, h, k)CSSs\) are denoted by \(S(n, h, k)\), \(K(n, h, k)\), and \(R(n, h, k)\) respectively. The sizes of minimal \((n, k)SSs\), \((n, k)KSSs\), or \((n, k)CSSs\) are denoted...
Example 1.1. \{123, 124, 135\}, \{123, 145, 246\} and \{123, 156, 246, 345\} are minimal \((6, 3)SSs\), \((6, 3)KSSs\) and \((6, 3)CSSs\) respectively.

Note that any collection of \(k\)-subsets of \([n]\) that is a superset of a \((n, k)SS\), \((n, k)KSS\) or \((n, k)CSS\) is also a \((n, k)SS\), \((n, k)KSS\) or \((n, k)CSS\) respectively.

Let \(\mathcal{S}_n\) denote the set of all minimal \((n)SSs\). Define \(V_{\text{min}}(\mathcal{S}_n) = \min_{S \in \mathcal{S}_n} \{V(S)\}\) and \(V_{\text{max}}(\mathcal{S}_n) = \max_{S \in \mathcal{S}_n} \{V(S)\}\). A \((n)SS\) which achieves \(V_{\text{min}}(\mathcal{S}_n)\) is said to be a minimum volume minimal \((n)SS\. A \((n)SS\) which achieves \(V_{\text{max}}(\mathcal{S}_n)\) is said to be a maximum volume minimal \((n)SS\. Similar definitions of minimum and maximum volume apply to \(KSSs\) and \(CSSs\) with \(\mathcal{K}_n\) and \(\mathcal{C}_n\) denoting the set of all minimal \((n)KSSs\) and all minimal \((n)CSSs\) respectively.

The number of non-isomorphic minimal \((n)KSSs\) and \((n)SSs\) is denoted by \(d_K(n)\) and \(d_S(n)\) respectively. A \(p\)-element in a collection of sets is an element which occurs in exactly \(p\) sets in the collection.

It is often convenient to represent the sets in a \(SS\) or \(KSS\) as rows in an array. In some of the representations of \(KSSs\) in this thesis extra spaces are left in some rows to highlight the structures of the \(KSSs\).

Further notation and definitions will be introduced as required in later chapters.
Chapter 2

Separating Systems and
Completely Separating Systems
Chapter 2. Separating Systems and Completely Separating Systems

2.1 Introduction

The main aim of this chapter is to present some historical developments in the study of separating systems and completely separating systems and to derive new results on \((n, k)\) CSSs. These results will be used in Chapter 7.

In 1961 Renyi [18] raised the problem of finding minimal separating systems in the context of solving certain problems in information theory. Since then several variants have been treated in [5], [9] and [26].

Renyi [18] has shown that

**Theorem 2.1 (Renyi [18]).**

\[
S(n) = \lceil \log_2 n \rceil.
\]

Katona [9] has shown that \(S(n, 1, k) = S(n, k)\) and

\[
\frac{n \log_2 n}{k \log_2 \frac{n}{k}} \leq S(n, 1, k) \leq \left\lceil \frac{\log_2 n}{\log_2 \frac{n}{k}} \right\rceil \frac{n}{k}.
\]

Wegener [26] has simplified the upper bound of Katona. Their results are combined in the following theorem.

**Theorem 2.2 ([9, 26]).**

1. If \(k \geq \frac{n}{2}\), then \(S(n, 1, k) = \lceil \log_2 n \rceil\),
2. If \(k < \frac{n}{2}\), then \(\frac{n \log_2 n}{k \log_2 \frac{n}{k}} \leq S(n, 1, k) \leq \left\lceil \frac{\log_2 n}{\log_2 \frac{n}{k}} \right\rceil \left(\lceil \frac{n}{k} \rceil - 1\right)\).

In 1969, Dickson [7] introduced CSSs and showed that \(R(n) \sim S(n)\). Spencer [25] found the exact value of \(R(n)\) for each \(n\) as stated in Theorem 2.4. He used the duality of CSSs and antichains and applied Sperner’s theorem (Theorem 2.3).

This is explained below.

**Lemma 2.1 (Spencer [25]).** A family \(C = \{A_1, A_2, \ldots, A_m\}\) on \(n\) is a completely separating system if and only if its dual \(C^* = \{X_1, X_2, \ldots, X_n\}\) is an antichain.
Theorem 2.3 (Sperner's Theorem [1, p. 2]). Let $A$ be an antichain of subsets of $[n]$. Then

$$|A| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$ 

Theorem 2.4 (Spencer [25]). $R(n) = \min \left\{ t : \binom{t}{\lfloor \frac{t}{2} \rfloor} \geq n \right\}$

Proof. Let $C = \{A_1, A_2, \ldots, A_t\}$ be a completely separating system on $[n]$. By Lemma 2.1, the dual of $C$ is an antichain of $n$ sets on $[t]$. By Theorem 2.3 such an antichain exists if and only if $\binom{t}{\lfloor \frac{t}{2} \rfloor} \geq n$. □

Spencer did not explicitly construct any $(n)$CSSs although his proof of Theorem 2.4 provides a method for doing this. In 1984, Cai used a construction based upon graphs to determine the exact value for $R(n, 1, k)$ when $n > \frac{k^2}{2}$ (Theorem 2.5). Cai did not provide example minimal $(n, 1, k)$CSSs although a method of construction is provided by the proof of Theorem 2.5. For the proof of Theorem 2.5, note that a simple graph is a graph which contains no self-loops and which has at most one edge between any pair of vertices.

Theorem 2.5 (Cai [4]). $R(n, 1, k) = \left\lceil \frac{2n}{k} \right\rceil$ for $n > \frac{k^2}{2}$.

Proof. Let $C = \{A_1, A_2, \ldots, A_m\}$ be a minimal completely separating system and $C^* = \{X_1, X_2, \ldots, X_n\}$ be its dual. Suppose $X_j = \{i\}$ is a singleton of $C^*$. As $C^*$ is an antichain with no other sets of $C^*$ contain $i$. Thus $A_i$ is a singleton of $C$. Relabel the sets if necessary so that the singletons have the smallest indices. That is, for some $l$, $1 \leq l \leq \min(n, m)$ $A_1, A_2, \ldots, A_l$ and $X_1, X_2, \ldots, X_l$ are all singletons. Then

$$\sum_{i=1}^{l} |A_i| + \sum_{i=l+1}^{m} |A_i| = \sum_{j=1}^{l} |X_j| + \sum_{j=l+1}^{n} |X_j|.$$ 

Hence

$$\sum_{i=l+1}^{m} |A_i| = \sum_{j=l+1}^{n} |X_j|.$$
Since $|A_i| \leq k$ and $|X_j| \geq 2$ for $i, j \geq l + 1$,

$$k(m - l) \geq \sum_{i=l+1}^{m} |A_i| = \sum_{j=l+1}^{n} |X_j| \geq 2(m - l).$$

Hence $km \geq 2n$. Therefore $R(n, 1, k) \geq \left\lceil \frac{2n}{k} \right\rceil$.

Next, it will be proved that $R(n, 1, k) \leq \left\lceil \frac{2n}{k} \right\rceil$ for $n > k^2/2$. Let $p = \left\lceil \frac{2n}{k} \right\rceil$ and $n > k^2/2$. Then $p \geq k + 1$. Now, construct a simple graph $G$ with vertex-set $V(G)$ and edge-set $E(G)$ such that $|V(G)| = p, |E(G)| = n, \Delta(G) \leq k$, where $\Delta(G)$ denotes the maximum degree of the vertices in $G$. Therefore every vertex of $G$ can be made incident, if necessary, with $k$ other vertices. Next, form

$$A_i = \{j : e_j \in E(G), e_j \text{ is incident with } v_i\}$$

for all $v_i \in V(G)$.

It will be proved that $\mathcal{C} = \{A_1, ..., A_p\}$ is a completely separating system on $E(G)$. Let $e_i$ and $e_j$ be any two edges of $G$. If $e_i$ and $e_j$ have no common vertex, then there is nothing to prove. If $e_i$ and $e_j$ are incident with vertex $v_i$, then both $e_i$ and $e_j$ are in $A_i$. Then, as $G$ is a simple graph, $e_i$ and $e_j$ are necessarily incident with two other distinct vertices and so belong to two different sets in $\mathcal{C}$. Hence any pair of edges is separated by a set of $\mathcal{C}$ and $\mathcal{C}$ is a completely separating system for $E(G)$. Therefore $R(n, 1, k) \leq \left\lceil \frac{2n}{k} \right\rceil$ for $n > k^2/2$. This completes the proof of the theorem.\qed

Designing $(n)CSSs$ which achieve $R(n)$ is not difficult by using the duality of CSSs and antichains. The process is described in [12]. Roberts [20] also determined the number of non-isomorphic $(n)CSSs$ which achieve $R(n)$ for $n \leq 10$. Consideration of $(n, h, k)CSSs$ for $h \neq k$ was introduced by Roberts [20]. $R(n, h, k)$ is known for all values of $h, k$ and $n$ with $1 \leq h \leq k < n \leq 10$ and for $n \geq \left\lceil \frac{k+1}{2} \right\rceil - 1$.

Ramsay et al. [16] introduced the notion of $(n,k)CSSs$. Some of their results
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needed for this thesis are presented below. The next lemma is a simple symmetry observation.

Lemma 2.2 (Ramsay et al. [16]). For all \( k, n, 1 \leq k < n, R(n, k) = R(n, n-k) \).

\[ \text{Proof.} \quad \text{Let } C \text{ be } (n,k)\text{CSS. Given any two distinct elements } i, j \in [n], \text{ there exists } A \in C \text{ such that } i \in A \text{ and } j \notin A. \text{ Hence } i \notin A' \text{ and } j \in A \text{ where } A' \text{ is the complement of } A. \text{ Thus } C' \text{ is a } (n, n-k)\text{CSS.} \]

Furthermore, if \( C \) is a minimal \((n,k)\text{CSS}\) then \( C' \) is a minimal \((n, n-k)\text{CSS}\).

To see this note that if \( C' \) is not minimal then there is a \((n, n-k)\text{CSS} \ D \) with \( |D| < |C| \). Then \( D' \) is a \((n,k)\text{CSS} \) with \( |D'| < |C| \) and this contradicts the minimality of \( C \).

Lemma 2.3 (Ramsay et al. [16]). If \( C \) is a \((n,k)\text{CSS} \) and \( 1 < k < n \), then every element of \([n]\) occurs at least twice in \( C \).

\[ \text{Proof.} \quad \text{Let } C \text{ be a } (n,k)\text{CSS with } 1 < k < n. \text{ It is clear that every element of } [n] \text{ must occur at least once in } C. \text{ Now suppose there exists } i \in [n] \text{ which occurs only once in } C, \text{ say in the set } A. \text{ As } |A| \geq 1, i \text{ can not be separated from the other elements of } A. \text{ This contradicts } C \text{ being a } (n,k)\text{CSS.} \]

The following lemma provides a lower bound for \( R(n,k) \).

Lemma 2.4 (Ramsay et al. [16]). For all \( k, n, 2 \leq k < n, R(n, k) \geq \left\lceil \frac{2n}{k} \right\rceil \).

\[ \text{Proof.} \quad \text{For } k \geq 2, \text{ every element of } [n] \text{ must occurs in at least two } k\text{-sets of any } (n,k)\text{CSS by Lemma 2.3. Hence } kR(n,k) \geq 2n \text{ which gives the required inequality as } R(n,k) \text{ is an integer.} \]

The next lemma is obvious.
Lemma 2.5 (Ramsay et al. [16]). For \( n \geq 2 \), \( R(n, 1) = n \).

The next theorem combines the main results from [16] and [17]. The proof of this theorem is omitted.

Theorem 2.6 (Ramsay et al. [16, 17]). For \( k < n \):

1. If \( n \geq \binom{k+1}{2} \) and \( k \geq 2 \), then \( R(n, k) = \left\lceil \frac{2n}{k} \right\rceil \);
2. If \( n = \binom{k+1}{2} - 1 \) and \( k \geq 3 \), then \( R(n, k) = k + 2 = \left\lceil \frac{2n}{k} \right\rceil + 1 \);
3. If \( k^2 \leq n \leq \binom{k+1}{2} - 2 \) and \( k \geq 5 \), then \( R(n, k) = k + 1 \) with \( R(n, k) = \left\lceil \frac{2n}{k} \right\rceil \) except \( n = \frac{k^2}{2} \);
4. If \( \binom{k}{2} \leq n < \frac{k^2}{2} \) and \( k \geq 5 \), then \( R(n, k) = k + 1 > \left\lceil \frac{2n}{k} \right\rceil \).

Note that \( R(n, k) \) is not monotonic in \( n \) for \( k \) fixed. For example, \( R(9, 4) = 6 \), \( R(10, 4) = 5 \) and \( R(11, 4) = 6 \).

### 2.2 Some New Results on \((n, k)CSSs\)

The following lemmas are needed to prove Theorem 7.5 which determines \( K(n, k) \) for each \( k \) except for a finite number of values of \( n \) in each case.

Lemma 2.6. For \( 2 \leq p < n \) and \( p \geq \frac{2n}{3} \),

\[
R(n - p, 1) = n - p \leq \frac{n}{2}.
\]

**Proof.** Suppose \( p \geq \frac{2n}{3} \). Then \( 2(n - p) \leq p \). Hence \( n - p \leq \frac{n}{2} \). By Lemma 2.5, \( R(n - p, 1) = n - p \). Therefore \( R(n - p, 1) = n - p \leq \frac{n}{2} \) as required. \( \square \)

Lemma 2.7. If \( n = \binom{k+1}{2} \), \( 2 < k < n \), then \( R(n - k, k - 1) = k \).

**Proof.** If \( n = \binom{k+1}{2} \) and \( 2 < n < n \) then \( n - k = \binom{k}{2} \). The lemma follows from Theorem 2.6(1) with \( k - 1 \) replacing \( k \) and \( n - k \) replacing \( n \) in the theorem.
Lemma 2.8. For all \( k, p \) and \( n \) with \( 2 < k < p < n \) and \((k + 1)(p - 1) < 2n \leq (k + 1)p\),

\[ R(n - p, k - 1) \leq p. \]

Proof. Suppose \( 2 < k < p < n \) and \((k + 1)(p - 1) < 2n \leq (k + 1)p\). Then

\[(k + 1)(p - 1) - 2p < 2n - 2p \leq (k + 1)p - 2p\]

so that

\[(k - 1)(p - 1) - 2 < 2(n - p) \leq (k - 1)p. \quad (2.1)\]

As \( p = k + i \) for some \( i \), \( 1 \leq i \leq n - k \), \((2.1)\) can be written as

\[(k - 1)(k + i - 1) - 2 < 2(n - p) \leq (k - 1)(k + i)\]

and hence

\[\frac{(k - 1)(k + i - 1)}{2} - 1 < n - p \leq \frac{(k - 1)(k + i)}{2}.\]

Since \( n - p \in \mathbb{Z}^+ \),

\[\frac{(k - 1)(k + i - 1)}{2} \leq n - p \leq \frac{(k - 1)(k + i)}{2}.\]

Set \( k' = k - 1 \) and \( n' = n - p \). Then

\[\frac{k'(k' + i)}{2} \leq n' \leq \frac{k'(k' + 1 + i)}{2}. \quad (2.2)\]
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and so for each $i \geq 1$,

$$n' \geq \frac{k'(k' + i)}{2} \geq \frac{k'(k' + 1)}{2}.$$ 

Hence

\[
R(n', k') = \left\lfloor \frac{2n'}{k'} \right\rfloor 
\leq \frac{k'(k' + 1 + i)}{k'} \quad \text{by (2.2)}
\]

\[
= k' + 1 + i
\]

\[
= k + i
\]

\[
= p.
\]

Therefore $R(n - p, k - 1) \leq p$ as required.
Chapter 3

Minimal \((n)\) and \((n, h, k)\) Covering Separating Systems
Chapter 3. Minimal \((n)\) and \((n, h, k)\) Covering Separating Systems

3.1 Introduction

This chapter considers \((n)KSSs\) and \((n, h, k)KSSs\). In Section 3.2 relationships between \((n)KSSs\) and \((n)SSs\) are established. These relationships help to determine the values of \(K(n)\) and \(K(n, h, k)\) derived in this section. Section 3.3 provides several characterisations of \((n)KSSs\) and \((n, 1, k)CSSs\) while Section 3.4 gives a characterisation of the duals of \((n)KSSs\).

3.2 Relationship between \((n)KSSs\) and \((n)SSs\) and Results for \(K(n)\) and \(K(n, h, k)\)

In this section the relationship between \((n)KSSs\) and \((n)SSs\) is investigated. It will help, among other things, to determine the value of \(K(n)\) in terms of \(S(n + 1)\) (Theorem 3.1) and in terms of \(S(n)\) (Theorem 3.2). The following lemma is derived from the definition of a \(KSS\).

**Lemma 3.1.** If \(\mathcal{K}\) is a \((n)KSS\) then \(\mathcal{K}\) is a \((n)SS\) and a \((n + 1)SS\).

**Note 3.1.** The converse is not always true as \(S = \{12, 13\}\) is a \((4)SS\) but not a \((4)KSS\).

**Theorem 3.1.** For all \(n \geq 2\), \(K(n) = S(n + 1)\).

**Proof.** Assume that \(\mathcal{K}\) is a minimal \((n)KSS\). Then \(n\) elements of \([n + 1]\) appear in \(\mathcal{K}\) and are separated in \(\mathcal{K}\). Therefore \(\mathcal{K}\) is a \((n + 1)SS\) so

\[
S(n + 1) \leq K(n). \tag{3.1}
\]

Assume that \(S\) is a minimal \((n + 1)SS\). If \(S\) contains exactly \(n\) elements of \([n + 1]\) then \(S\) is a \((n)KSS\). If \(S\) contains \(n + 1\) elements of \([n + 1]\) then the removal of
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the element \(n + 1\) from each set in which it occurs in \(S\) gives a collection of sets which is a \((n)KSS\). Therefore

\[
K(n) \leq S(n + 1).
\]

Inequalities (3.1) and (3.2) give the result.

Theorem 3.1 has the following three corollaries.

**Corollary 3.1.** If \(K\) is a minimal \((n)KSS\) then \(K\) is a minimal \((n + 1)SS\).

**Proof.** Let \(K\) be a minimal \((n)KSS\). As \(K\) is a \(KSS\) on \([n]\), \(K\) is also a \(SS\) on \([n]\) by Lemma 3.1. Since \(K\) contains \(n\) elements, \(K\) is a \(SS\) on \([n + 1]\) in which the element \(n + 1\) does not occur. By Theorem 3.1, \(K(n) = S(n + 1)\) so \(K\) is a minimal \((n + 1)SS\).

**Note 3.2.** The converse does not apply. For example, \(S = \{123, 145, 246\}\) is a minimal \((6)SS\) but not a minimal \((5)KSS\).

The two corollaries below give exact formulas for \(K(n)\) for all \(n \geq 2\) in terms of \(n\) rather than \(S(n + 1)\).

**Corollary 3.2.** For all \(n \geq 2\), \(K(n) = \lceil \log_2 (n + 1) \rceil \).

**Proof.** This follows immediately from Theorem 3.1 and Theorem 2.1.

**Corollary 3.3.** For all \(n \geq 2\), \(K(n) = \lfloor \log_2 2n \rfloor \).
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Proof. Let \( p \in \mathbb{Z}^+ \). Assume \( n = 2^p \). Then

\[
[log_2 2n] = [log_2 2^{p+1}]
= p + 1
= [log_2 n] + 1
= [log_2(n + 1)]
= K(n)
\]

by Corollary 3.2.

Assume \( 2^{p-1} < n < 2^p \). Hence \( 2^p < 2n < 2^{p+1} \). Then

\[
[log_2 2n] = p
= [log_2 n]
= [log_2(n + 1)]
= K(n)
\]

by Corollary 3.2.

In both cases, \( K(n) = [log_2 2n] \).

\[ \square \]

Note 3.3. By Corollary 3.3, \( K(n) < n \) for \( n \geq 3 \).

Theorem 3.1 expressed \( K(n) \) in terms of \( S(n + 1) \). Theorem 3.2 expresses \( K(n) \) in terms of \( S(n) \).

Theorem 3.2. For all \( n \geq 2 \),

\[
K(n) = \begin{cases} 
S(n) + 1 & \text{if } n = 2^p, p \in \mathbb{Z}^+ \\
S(n) & \text{otherwise}
\end{cases}
\]

Proof. Assume that \( n = 2^p, p \in \mathbb{Z}^+ \). By Theorem 2.1, \( S(n) = [log_2 n] = p \) as \( n = 2^p \). By Theorem 3.1, \( K(n) = S(n + 1) = [log_2(n + 1)] = p + 1 \) as \( n = 2^p \).

Thus \( K(n) = S(n) + 1 \) in this case.

Assume \( n \neq 2^p, p \in \mathbb{Z}^+ \). Then \( [log_2 n] = [log_2(n + 1)] \). By Theorem 2.1, \( S(n) = [log_2 n] \), so that \( K(n) = S(n) \) in this case. \[ \square \]
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Theorem 3.2 states that \(K(n) = S(n)\) unless \(n\) is a power of 2. To illustrate this relationship, values of \(K(n)\) and \(S(n)\) for \(2 \leq n \leq 17\) are given in Table 3.1.

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
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<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>(K(n))</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 3.1: Known values of \(K(n)\) compared to \(S(n)\) for \(1 < n \leq 17\)

Theorem 3.2 has five corollaries.

**Corollary 3.4.** For all \(n \geq 2\), \(S(n) \leq K(n) \leq S(n) + 1\).

As any \((n, k)KSS\) is a \((n, h, k)KSS\) and any \((n, h, k)KSS\) is a \((n, k)KSS\) it is easy to see that for all \(n, h, k \in \mathbb{Z}^+\) with \(1 \leq h \leq k < n\), \(K(n) \leq K(n, h, k) \leq K(n, k)\). Combining this and the left inequality of Corollary 3.4 gives the following corollary.

**Corollary 3.5.** \(S(n) \leq K(n) \leq K(n, h, k) \leq K(n, k)\) for all \(n, h, k \in \mathbb{Z}^+\) with \(1 \leq h \leq k < n\).

**Corollary 3.6.** For all \(n \geq 2\), \(K(n) \leq K(n + 1) \leq K(n) + 1\).

*Proof.* Assume \(n = 2^p\). Then, by Theorem 3.2,

\[
K(n) = S(n) + 1 = p + 1, \tag{3.3}
\]

\[
K(n + 1) = S(n + 1) = p + 1. \tag{3.4}
\]

Assume \(n \neq 2^p\) and \(n + 1 = 2^p\). Then, by Theorem 3.2,

\[
K(n) = S(n) = p, \tag{3.5}
\]

\[
K(n + 1) = S(n + 1) + 1 = p + 1. \tag{3.6}
\]
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Assume \(n \neq 2^p\) and \(n + 1 \neq 2^p\). Then, by Theorem 3.2,

\[
K(n) = S(n) = p, \tag{3.7}
\]
\[
K(n + 1) = S(n + 1) = p. \tag{3.8}
\]

Combining (3.3), (3.4), (3.5), (3.6), (3.7), and (3.8) gives the result.

**Corollary 3.7.** For all \(p, n\) and \(n' \in \mathbb{Z}^+\) with \(2^{p-1} < n \leq n' < 2^p\), \(K(n) = K(n')\) and \(K(2^p) = K(n) + 1\).

**Proof.** It follows from Theorem 3.2 that \(K(n) = K(n')\) as \(S(n) = S(n')\) and \(K(2^p) = p + 1 = \lceil \log_2 n \rceil + 1 = K(n) + 1\).

The last corollary of Theorem 3.2 says that every minimal \((n)KSS\) is a minimal \((n)SS\) when \(n\) is not a power of 2. This is to be compared with Corollary 3.1.

**Corollary 3.8.** For all \(p \in \mathbb{Z}^+, n \neq 2^p\), if \(\mathcal{K}\) is a minimal \((n)KSS\) then \(\mathcal{K}\) is a minimal \((n)SS\).

**Proof.** For all \(p \in \mathbb{Z}^+\), let \(\mathcal{K}\) be a minimal \((n)KSS\). As \(\mathcal{K}\) is a \(KSS\) on \([n]\), \(\mathcal{K}\) is also a \(SS\) on \([n]\) by Lemma 3.1. By Theorem 3.2, \(K(n) = S(n)\) for \(n \neq 2^p\) so \(\mathcal{K}\) is a minimal \((n)SS\).

**Note 3.4.** The converse is not true. For example, \(\mathcal{S} = \{12, 13, 4\}\) is a minimal \((5)SS\) but not a minimal \((5)KSS\).

**Theorem 3.3.** For \(n \geq 2\):

1. if \(n \neq 2^p\), then \(\mathcal{S}\) is a minimal \((n)SS\) if and only if \(\mathcal{S}\) is a minimal \((n)KSS\) or a minimal \((n - 1)KSS\);
2. if \(n = 2^p\), then \(\mathcal{S}\) is a minimal \((n)SS\) if and only if \(\mathcal{S}\) is a minimal \((n - 1)KSS\).
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**Proof.** By Lemma 3.1, a \((n)KSS\) is a \((n)SS\) and a \((n + 1)SS\). Clearly, a \((n)SS\) or \((n + 1)SS\) which contains \(n\) elements is a \((n)KSS\). By Corollary 3.1, each minimal \((n)KSS\) is a minimal \((n + 1)SS\). By Corollary 3.8, each minimal \((n)KSS\) is a minimal \((n)SS\) for \(n \neq 2^p\). Conversely, it is clear that a minimal \((n)SS\) containing \(n\) elements is also a minimal \((n)KSS\) and that a minimal \((n + 1)SS\) containing \(n\) elements is also a minimal \((n)KSS\). Theorem 3.3 follows from these observations.

**Note 3.5.** Theorem 3.3 provides the precise relationship between minimal \((n)KSSs\) and minimal \((n)SSs\). This formalises the idea alluded to in the introduction that results on minimal \((n)KSSs\) can be used to derive results on minimal \((n)SSs\) and vice versa.

The last result in this section relates \(K(n + 1, 1, k)\) to \(K(n, 1, k)\).

**Lemma 3.2.** \(K(n + 1, 1, k) \geq K(n, 1, k)\) for all \(n, k \in \mathbb{Z}^+\) with \(1 \leq k \leq n\).

**Proof.** Let \(\mathcal{K}\) be a minimal \((n + 1, 1, k)KSS\). Remove all occurrences of the element \(n + 1\) from the members of \(\mathcal{K}\). Then the remaining \(n\) elements are still separated in at most \(|\mathcal{K}|\) sets. Hence the inequality follows.

**Note 3.6.** Lemma 3.2 implies that for all \(1 \leq k < n\), \(K(n, 1, k)\) is monotonic in \(n\) for fixed \(k\).

### 3.3 Characteristics of Minimal \((n)KSSs\) and Minimal \((n, 1, k)KSSs\)

In this section some characteristics of minimal \((n)KSSs\) and \((n, 1, k)KSSs\) are given. They will be necessary when cataloguing \((n)KSSs\) and \((n, 1, k)KSSs\) in Chapter 6. Characterisations of minimal \((n)KSSs\) and \((n, 1, k)KSSs\) are made
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in terms of the set \([n]\), of singleton sets, and of the number of \(p\)-elements, \(p \in \mathbb{Z}^+\),
in the covering separating systems.

**Lemma 3.3.** In a minimal \((n)\)KSS the set \([n]\) can only be included when \(n = 2^p\),
\(p \in \mathbb{Z}^+\).

**Proof.** Assume \(n = 2^p\), \(p \in \mathbb{Z}^+\). Then \(K(n) = S(n) + 1\) by Theorem 3.2. Hence
the set \([n]\) can be included in a minimal \((n)\)SS to form a minimal \((n)\)KSS.

Let \(n \neq 2^p\), \(p \in \mathbb{Z}^+\) and let \(\mathcal{K}\) be a minimal \((n)\)KSS. As \(K(n) = S(n)\) by
Theorem 3.2 so the elements of \([n]\) can be not separated in the remaining sets of \(\mathcal{K}\). Hence \([n]\) is not in \(\mathcal{K}\).

This proves the lemma. \(\Box\)

**Note 3.7.** Unlike the case for \((n)\)CSSs the set \([n]\) can be included in a minimal
\((n)\)KSS.

**Lemma 3.4.** (1) Let \(\mathcal{K}\) be a minimal \((n)\)KSS with \(K(m) = K(n)\) for some
\(m < n\). Then \(|A| \leq m\) for each \(A \in \mathcal{K}\).

(2) Let \(\mathcal{K}\) be a minimal \((n, 1, k)\)KSS with \(K(m, 1, k) = K(n, 1, k)\) for some
\(m < n\). Then \(|A| \leq m\) for each \(A \in \mathcal{K}\).

**Proof.** (1) Assume \(A \in \mathcal{K}\), \(|A| > m\). Then, as \(K(m) = K(n)\), the \(m + 1\) or more
elements of \(A\) cannot be separated in the remaining sets of \(\mathcal{K}\).

(2) The proof is similar to (1). \(\Box\)

**Lemma 3.5.** (1) If \(K(n + 1) = K(n)\), then there is no minimal \((n + 1)\)KSS
which contains a singleton set.

(2) If \(K(n + 1, 1, k) = K(n, 1, k)\), then there is no minimal \((n + 1, 1, k)\)KSS
which contains a singleton set.
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Proof. (1) Assume that there is a minimal \((n+1)KSS\) \(\mathcal{K}\) containing a singleton set, say \(\{n+1\}\). As \(\mathcal{K}\) is a \((n+1)KSS\) so \(\mathcal{K} \setminus \{\{n+1\}\}\) must be a \((n)KSS\). This is a contradiction if \(K(n+1) = K(n)\).

(2) The proof is similar to (1). \(\Box\)

Lemma 3.6. There is a minimal \((2)KSS\) which contains two singleton sets.

Proof. This is obvious since \(\mathcal{K} = \{\{1\}, \{2\}\}\) is a \((2)KSS\) and two elements cannot be covering separated in one set. \(\Box\)

Lemma 3.7. (1) If \(K(n+1) = K(n) + 1\), then there is a minimal \((n+1)KSS\) which contains a singleton set.

(2) If \(K(n+1, 1, k) = K(n, 1, k) + 1\), then there is a minimal \((n+1, 1, k)KSS\) which contains a singleton set.

Proof. (1) Let \(\mathcal{K}\) be a minimal \((n)KSS\). Then \(\mathcal{K} \cup \{\{n+1\}\}\) is a \((n + 1)KSS\). As \(K(n+1) = K(n) + 1\) so \(\mathcal{K} \cup \{\{n+1\}\}\) is a minimal \((n+1)KSS\).

(2) The proof is similar to (1). \(\Box\)

Combining Corollary 3.7, Lemma 3.5(1) and Lemma 3.7(1) gives the following Theorem.

Theorem 3.4. (1) A minimal \((2^p)KSS\) with \(p \geq 2\) contains at most one singleton set.

(2) A minimal \((n)KSS\) with \(n \neq 2^p\), \(p \in Z^+\) contains no singleton set.

The next lemma characterises \((n)KSS\)s in terms of the number of \(p\)-elements, \(p \in Z^+,\) they contain.

Lemma 3.8. Let \(\mathcal{K}\) be a \((n)KSS\) with \(|\mathcal{K}| = K\). Then:

(1) there is at most one 1-element in each set in \(\mathcal{K}\);
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(2) there are at most \((K - 1)\) 2-elements in each set in \(\mathcal{K}\);

(3) there are at most \(\frac{K(K-1)}{2}\) 2-elements in \(\mathcal{K}\).

Proof. (1) Assume that a set in \(\mathcal{K}\) contains more than one 1-element. Then none of these 1-elements can occur in any other set in \(\mathcal{K}\) and this contradicts \(\mathcal{K}\) being a \((n)KSS\).

(2) Assume that \(\mathcal{K}\) contains \(t\) 2-elements in one set. Each of these 2-elements must reoccur exactly once more in \(\mathcal{K}\). If \(t > |\mathcal{K}| - 1\) then at least two of the 2-elements must appear again in the same set in \(\mathcal{K}\) and this contradicts \(\mathcal{K}\) being a \((n)KSS\).

(3) This follows from using the pigeon-hole principle, together with (2) and the fact that \(\mathcal{K}\) contains exactly \(K\) sets. 

3.4 The Duals of \((n)KSS\)s

In this section the duals of \((n)KSS\)s are investigated. Their characterisation (Theorem 3.5 below) will prove useful when deriving results about the volumes of \((n)KSS\)s in Chapter 4 and when cataloguing \((n)KSS\)s in Chapter 6.

Theorem 3.5. Let \(\mathcal{K}\) be a collection of subsets of \([n]\). \(\mathcal{K}\) is a \((n)KSS\) of size \(K\) if and only if \(\mathcal{K}^*\), the dual of \(\mathcal{K}\), is a collection of \(n\) distinct non-empty subsets of \([K]\).

Proof. Assume \(\mathcal{K}\) is a \((n)KSS\) of size \(K\). Then, for any two distinct elements \(i,j \in [n]\), there exists \(A_p \in \mathcal{K}\) in which \(i \in A_p\) and \(j \notin A_p\). Thus \(p \in B_i \in \mathcal{K}^*\) and \(p \notin B_j \in \mathcal{K}^*\). Hence \(B_i \neq B_j\). It is clear that no \(B_i\) is the empty set since each element \(i \in [n]\) occurs at least once in \(\mathcal{K}\). Therefore \(\mathcal{K}^*\) is a collection of \(n\) distinct non-empty subsets of \([K]\).
Conversely, assume $\mathcal{K}^*$ is a collection of $n$ distinct non-empty subsets of $[K]$. Since each $B_i$ is non-empty, each element $i \in [n]$ occurs at least once in $\mathcal{K}$. Further, for any two distinct elements $i, j \in [n]$, there are $B_i, B_j \in \mathcal{K}^*$ with $B_i \neq B_j$. Hence $i$ and $j$ do not occur in exactly the same set in $\mathcal{K}$. Therefore $\mathcal{K} = \{A_1, A_2, \ldots, A_K\}$ is a $(n)KSS$.

**Note 3.8.** Let $\mathcal{K}^*$ be as in Theorem 3.5 and let $\mathcal{K}_1^*$ be the collection obtained from $\mathcal{K}^*$ by interchanging the order of two member $B_i$ and $B_j$ of $\mathcal{K}^*$. It is easy to see that the minimal $(n)KSS$ $\mathcal{K}$ corresponding to $\mathcal{K}^*$ and the minimal $(n)KSS$ $\mathcal{K}_1$ corresponding to $\mathcal{K}_1^*$ are isomorphic. In other words, $\mathcal{K}_1$ can be obtained from $\mathcal{K}$ by permuting the elements $i$ and $j$ of $[n]$ in $\mathcal{K}$. When $(n)KSS$s are considered later it is generally convenient to deal with representatives of the isomorphic classes.

Incidentally, Theorem 3.5 leads to another formula for $K(n)$ for all $n \geq 2$.

**Corollary 3.9.** For all $n \geq 2$, $K(n) = \min\{K : 2^K - 1 \geq n\}$.

**Proof.** Let $\mathcal{K}$ be a minimal $(n)KSS$ of size $K$. By Theorem 3.5, the dual collection $\mathcal{K}^*$ contains exactly $n$ distinct non-empty subsets of $[K]$. Hence $2^K - 1 \geq n$ since there are at most $2^K - 1$ non-empty subsets of $[K]$. As $K$ is the least positive integer with $2^K - 1 \geq n$, $K(n) = K$. This proves the corollary.  $\square$
Chapter 4

Minimal \((n)\) Covering Separating Systems and Their Volumes
4.1 Introduction

The purpose of this chapter is to present results about the minimum volume of minimal \((n)\text{KSSs}\). These results are useful when cataloguing all non-isomorphic \((n)\text{KSSs}\) which achieve \(K(n)\) for \(2 \leq n \leq 7\) in Chapter 6. Results on the minimum volume of minimal \((n)\text{KSSs}\) are also useful to determine some values of \(K(n, k)\) in Chapter 8.

Results on volumes are obtained by a further characterisation of minimal \((n)\text{KSSs}\) and their duals. This is done in Section 4.2. This same characterisation of the duals of minimal \((n)\text{KSSs}\) leads to results about the number of labelled and unlabelled minimum volume minimal \((n)\text{KSSs}\). This is done in Section 4.3.

4.2 Volumes of Minimal \((n)\text{KSSs}\)

In this section further characterisations of minimal \((n)\text{KSSs}\) and their duals help derive several results on volumes of minimal \((n)\text{KSSs}\). The following simple theorem is used to derive a lower bound on the minimum volume of minimal \((n)\text{KSSs}\).

**Theorem 4.1.** In any minimal \((n)\text{KSS}, there are at most \(K(n)\) elements which occur exactly once.

**Proof.** Let \(\mathcal{K}\) be a minimal \((n)\text{KSS}. Assume that there are more than \(K(n)\) elements of \([n]\) which occur exactly once in \(\mathcal{K}\). Then there are at least two of the elements which occur together in a member of \(\mathcal{K}\). Thus these two elements are not separated from each other. This contradicts \(\mathcal{K}\) being a separating system. \(\square\)

**Corollary 4.1.** In any minimal \((n)\text{KSS}, there are at least \(n - K(n)\) elements which occur at least twice.
Corollary 4.2. Let \( \mathcal{K} \) be a minimal \((n)KSS\). Then \( V(\mathcal{K}) \geq 2n - |\mathcal{K}| \).

Proof. If \( \mathcal{K} \) is a minimal \((n)KSS\) then each element of \([n]\) occurs at least once in \( \mathcal{K} \) and by Corollary 4.1, at least \( n - |\mathcal{K}| \) elements occur at least twice in \( \mathcal{K} \). Therefore \( V(\mathcal{K}) \geq 2(n - |\mathcal{K}|) + |\mathcal{K}| = 2n + |\mathcal{K}|. \)

Consideration of the bounds on the volume of a \((n, h, k)KSS\) yields the following lemma.

Lemma 4.1. Let \( \mathcal{K} \) be a \((n, h, k)KSS\). Then \( h|\mathcal{K}| \leq V(\mathcal{K}) \leq k|\mathcal{K}|. \)

Results about minimum and maximum volumes of \((n)KSSs\) are now derived.

The idea behind the next theorem is that if \( K(n) = K \) then choosing the \( n \) smallest non-empty subsets of \([K]\) provides the dual of a minimum volume minimal \((n)KSS\) by application of Theorem 3.5.

Theorem 4.2. Assume \( K(n) = K \), and let \( r = \max_{t \in [K]} \{ \sum_{i=1}^{r} \binom{K}{i} \leq n \} \) and \( p = n - \sum_{i=1}^{r} \binom{K}{i} \). Then \( V_{\min}(\mathcal{K}_n) = \sum_{i=1}^{r} i \binom{K}{i} + p(r + 1) \).

Proof. Let \( \mathcal{K} = \{A_1, A_2, ..., A_K\} \) be a minimal \((n)KSS\). By Theorem 3.5 the dual of \( \mathcal{K}^* \) of \( \mathcal{K} \) is a collection \( \{B_1, B_2, ..., B_n\} \) of subsets of \([K]\) with each \( B_i \neq \emptyset \) and \( \sum_{i=1}^{n} |B_i| = \sum_{j=1}^{K} |A_j| \). So if \( \sum_{i=1}^{n} |B_i| \) is minimum over all choices of \( n \) distinct non-empty subsets of \([K]\) then \( \sum_{j=1}^{K} |A_j| \) is minimum over all choices of \((n)KSSs\) in \( K \) sets. Clearly \( \sum_{i=1}^{n} |B_i| \) is minimum when \( \mathcal{K}^* = \{B_i, 1 \leq i \leq n\} \) consist of all the \( i \)-sets, \( 1 \leq i \leq r \), and of sufficient \((r + 1)\)-sets to ensure that if \( r = \max_{t \in [K]} \{ \sum_{i=1}^{r} \binom{K}{i} \leq n \} \) and \( p = n - \sum_{i=1}^{r} \binom{K}{i} \) then \( V(\mathcal{K}^*) = \sum_{i=1}^{r} i \binom{K}{i} + p(r + 1) \). Therefore \( V_{\min}(\mathcal{K}_n) = \sum_{i=1}^{r} i \binom{K}{i} + p(r + 1) \). \( \square \)

Corollary 4.3. If \( \mathcal{K} \) is a minimum volume minimal \((n)KSS\) with \( K(n) = K \) then there exist \( K \) elements of \([n]\) which occur exactly once in \( \mathcal{K} \).
Chapter 4. Minimal (n) Covering Separating Systems and Their Volumes

**Proof.** By Note 3.3 and by Theorem 4.2, the dual $K^*$ of $K$ must contain at least all the singleton sets of $[K]$. By the definition of the dual, each element of $[K]$ must occur exactly once in $K$. This proves the corollary. \(\square\)

**Note 4.1.** The converse is not true. For example, $K = \{12, 13, 14\}$ is a minimal $(4)KSS$ in which there are three elements (2, 3 and 4) which occur once in $K$ but $K$ is not a minimum volume minimal $(4)KSS$ since a minimum volume minimal $(4)KSS$ is $\{12, 13, 4\}$.

The following theorem is similar to Theorem 4.2 but applies to maximum volume minimal $(n)KSS$. It shows that choosing the $n$ largest non-empty subsets of $[K]$ provides the dual of a maximum volume minimal $(n)KSS$ by an application of Theorem 3.5.

**Theorem 4.3.** Assume $K(n) = K$, and let $r' = \max_{0 \leq r' \leq K-1} \left\{ \sum_{i=0}^{r'} \binom{K}{i} \leq n \right\}$ and $p' = n - \sum_{i=0}^{r'} \binom{K}{i}$. Then $V_{\text{max}}(K_n) = \sum_{i=0}^{r'} i \binom{K}{i} + p'(K - r' - 1)$.

**Proof.** Let $K = \{A_1, A_2, ..., A_K\}$ be a minimal $(n)KSS$. By Theorem 3.5 the dual of $K^*$ of $K$ is a collection $\{B_1, B_2, ..., B_n\}$ of subsets of $[K]$ with each $B_i \neq \emptyset$ and $\sum_{i=1}^{n} |B_i| = \sum_{j=1}^{K} |A_j|$. So if $\sum_{i=1}^{n} |B_i|$ is maximum over all choices of $n$ distinct non-empty subsets of $[K]$ then $\sum_{j=1}^{K} |A_j|$ is maximum over the choices of $(n)KSS$s in $K$ sets. Clearly $\sum_{i=1}^{n} |B_i|$ is maximum when $K^* = \{B_i, 1 \leq i \leq n\}$ consists of all the $i$-sets, $K - r' \leq i \leq K$, and of sufficient $(K - r' - 1)$-sets to ensure that if $r' = \max_{0 \leq r' \leq K-1} \left\{ \sum_{i=0}^{r'} \binom{K}{i} \leq n \right\}$ and $p' = n - \sum_{i=0}^{r'} \binom{K}{i}$ then $K^* = \sum_{i=0}^{r'} i \binom{K}{i} + p'(K - r' - 1)$. Therefore $V_{\text{max}}(K_n) = \sum_{i=0}^{r'} i \binom{K}{i} + p'(K - r' - 1)$. \(\square\)

**Corollary 4.4.** If $K$ is a maximum volume minimal $(n)KSS$ then there is an element of $[n]$ which occurs in every set in $K$.

**Proof.** By Theorem 4.3, the dual $K^*$ of $K$ must contain the set $[K]$. By the definition of the dual, there is an element which must be in each $A_i \in K$, $1 \leq
4.3 Number of Minimum Volume Minimal \((n)KSSs\)

Using results from Section 4.2 it is possible to count the number of labelled minimum (maximum) volume minimal \((n)KSSs\). This is done in Theorem 4.4. A partial result on the number of unlabelled minimum (maximum) volume minimal \((n)KSSs\) is also given.

**Theorem 4.4.** (1) Let \(r, p\) and \(K\) be as in Theorem 4.2. The number of labelled minimum volume minimal \((n)KSSs\) is 
\[
N = \binom{K}{r+1}\binom{r+1}{p}
\]
(2) Let \(r', p'\) and \(K\) be as in Theorem 4.3. The number of labelled maximum volume minimal \((n)KSSs\) is 
\[
N' = \binom{K-r'-1}{p'}
\]

**Proof.** (1) As \(p\) subsets of size \(r + 1\) of \([K]\) can be chosen from the \(\binom{K}{r+1}\) subsets size \(r + 1\), 
\[
N = \binom{K}{r+1}\binom{r+1}{p}
\]

(2) As \(p'\) subsets of size \(K - r' - 1\) of \([K]\) can be chosen from the \(\binom{K}{K-r'-1}\) subsets size \(K - r' - 1\), 
\[
N' = \binom{K-r'-1}{p'}
\]

**Note 4.3.** If \(p = 0\) (respectively, \(p' = 0\)) then there is a unique minimum (respectively, maximum) volume minimal \((n)KSS\).

**Lemma 4.2.** (1) Let \(r, p\) and \(K\) be as in Theorem 4.2. If \(p = 1\) or \(p = \binom{K}{r+1} - 1\) then there is a unique minimum volume minimal \((n)KSS\).
(2) Let $r', p'$ and $K$ be as in Theorem 4.3. If $p' = 1$ or $p' = (K - r - 1) - 1$ then there is a unique maximum volume minimal $(n)KSS$.

Proof. (1) Assume $K(n) = K$, $p = 1$ and $n = \sum_{i=1}^{r} \binom{K}{i} + 1$. By Theorem 3.5, the dual of any minimal $(n)KSS$ is a collection of $n$ non-empty subsets of $[K]$. Thus each dual contains all non-empty $i$-subsets for all $1 \leq i \leq r$ and one non-empty $(r + 1)$-subset of $[K]$. There are $\binom{K}{r+1}$ choices for the non-empty $(r + 1)$-subsets to be included in the dual but all are isomorphic. Thus there is a unique non-isomorphic minimum volume dual of $n$ non-empty subsets of $[K]$. Therefore there is a unique non-isomorphic minimum volume minimal $(n)KSS$.

Assume $K(n) = K$, $p = (\binom{K}{r+1}) - 1$ and $n = \sum_{i=1}^{r} \binom{K}{i} + \binom{K}{r+1} - 1$. By Theorem 3.5, the dual of any minimal $(n)KSS$ is a collection of $n$ non-empty subsets of $[K]$. Thus each dual contains all non-empty $i$-subsets for all $1 \leq i \leq r$ and all non-empty $(r + 1)$-subset but one of the non-empty $(r + 1)$-subsets of $[K]$. There are $\binom{K}{r+1}$ choices for the non-empty $(r + 1)$-subsets to be excluded from the dual but all are isomorphic. Thus there is a unique non-isomorphic minimum volume dual of $n$ non-empty subsets of $[K]$. Therefore there is a unique non-isomorphic minimum volume minimal $(n)KSS$.

(2) The proof is similar to the proof (1) so its details are not included. □

The section concludes with the consideration of the case when the unique minimal volume minimal $(n)KSS$ is also the unique maximal volume minimal $(n)KSS$.

Theorem 4.5. $V_{\text{min}}(\mathcal{K}_n) = V_{\text{max}}(\mathcal{K}_n)$ if and only if $n = 2^K - 1$

Proof. Assume $V_{\text{min}}(\mathcal{K}_n) = V_{\text{max}}(\mathcal{K}_n)$. Then $\mathcal{K}^*$ must contain all of non-empty subsets of $[K]$. Hence $n = 2^K - 1$.

Conversely, let $\mathcal{K}$ be a minimal $(n)KSS$. Assume $n = 2^K - 1$. By Corollary 3.2, $K(n) = K$. Since there are only $2^K - 1$ non-empty subsets of $[K]$, $\mathcal{K}^*$, the dual
of \( \mathcal{K} \), contains all of these non-empty subsets of \([K]\). By Theorem 4.2 and 4.3, 
\[ V_{\text{min}}(\mathcal{K}_n) = V_{\text{max}}(\mathcal{K}_n). \]

**Corollary 4.5.** There is a unique minimal \((n)KSS\) with \(K(n) = K\) if and only if \(n = 2^K - 1\).

*Proof.* There is only one collection of non-empty subsets of \([K]\) with size \(n\) so the dual of this collection must be a unique minimal \((n)KSS\) and \(K(n) = K\).

Conversely, assume that there is a unique minimal \((n)KSS\) with \(K(n) = K\). Then \(V_{\text{min}}(\mathcal{K}_n) = V_{\text{max}}(\mathcal{K}_n)\). Hence, by Theorem 4.5, \(n = 2^K - 1\). \(\square\)

**Corollary 4.6.** \(n = 2^K - 1\) if and only if \(K(n) = K(n, 2^{K-1})\).

*Proof.* Assume \(n = 2^K - 1\). Then \(K(n) = K\) by Corollary 4.5. In this case \(\mathcal{K}^*\), the dual of \(\mathcal{K}\), consists of all non-empty subsets of \([K]\). Each element \(i\) of \([K]\) occurs in \(2^{K-1}\) of these subsets. As there are \(2^{K-1} - 1\) subsets of \([K]\) which do not contain \(i\), the dual of \(\mathcal{K}^*\) is a \((n, 2^{K-1})KSS\). By Corollary 3.5, 
\[ K(n, 2^{K-1}) = K(n). \]

The converse is obvious. \(\square\)
Chapter 5

Minimal \((n)\) Separating Systems and Their Volumes
Chapter 5. Minimal \((n)\) Separating Systems and Their Volumes

5.1 Introduction

The only difference between a \((n)KSS\) and a \((n)SS\) is that in the latter there is no requirement that each element in \([n]\) occurs in the \((n)SS\). In the same manner as in Chapter 4, results about volumes of minimal \((n)SSs\) are derived from the characterisation of \((n)SSs\) and their duals. This is done in Section 5.2. Results about the number of labelled and unlabelled minimum volume minimal \((n)SSs\) are given in Section 5.3.

Results about volumes of minimal \((n)SSs\) are used when determining the number of non-isomorphic minimal \((n)SSs\) for \(n \leq 8\) in Chapter 6.

5.2 Volumes of Minimal \((n)SSs\)

Theorem 5.1 gives a characterisation of \((n)SSs\) and Theorem 5.2 characterises their duals.

**Theorem 5.1.** Let \(S\) be a minimal \((n)SS\). Then:

1. there is at most one element of \([n]\) which is not in \(S\);
2. there are at most \(S(n)\) elements which occur exactly once in \(S\).

**Proof.** (1) This follows from the definition of a separating system.

(2) The proof is similar to the proof of Theorem 4.1 and so its details are not included.

There are two corollaries of Theorem 5.1.

**Corollary 5.1.** In any minimal \((n)SS\), there are at least \(n - S(n) - 1\) elements which occur at least twice.

**Corollary 5.2.** Let \(S\) be a minimal \((n)SS\). Then \(V(S) \geq 2(n - 1) - |S|\).
Chapter 5. Minimal \((n)\) Separating Systems and Their Volumes

Proof. By Corollary 5.1, \(V(S) \geq 2(n - |S| - 1) + |S| = 2(n - 1) - |S|\).

Theorem 5.2. Let \(S\) be a collection of subsets of \([n]\). \(S\) is a \((n)SS\) of size \(S\) if and only if \(S^*\), the dual of \(S\), is a collection of \(n\) distinct subsets of \([S]\).

Proof. The proof is similar to the proof of Theorem 3.5 and so its details are not included.

Note 5.1. (1) Unlike the case for \((n)KSSs\) the empty set is allowed in the collection \(S^*\) in Theorem 5.2.

(2) Let \(S^*\) be as in Theorem 5.2 and let \(S_i^*\) be the collection obtained from \(S^*\) by interchanging the order of two members \(B_i\) and \(B_j\) of \(S^*\). It is easy to see that the minimal \((n)SS\) \(S\) corresponding to \(S^*\) and the minimal \((n)SS\) \(S_i\) corresponding to \(S_i^*\) are isomorphic. That is, \(S_i\) can be obtained from \(S\) by permuting the elements \(i\) and \(j\) of \([n]\) in \(S\). Thus when \((n)SSs\) are considered later it is generally convenient to deal with representatives of the isomorphic classes.

The following corollary of Theorem 5.2 gives an alternative derivation of the value of \(S(n)\).

Corollary 5.3. For \(n \geq 2\), \(S(n) = \min\{S : 2^S \geq n\}\).

Proof. Let \(S\) be a minimal \((n)SS\) of size \(S\). By Theorem 5.2, the collection \(S^*\) contains exactly \(n\) distinct subsets of \([S]\). Hence \(2^S \geq n\) since there are at most \(2^S\) subsets of \([S]\). As \(S\) is the least positive integer with \(2^S \geq n\), then \(S(n) = S\).

This proves the corollary.

The remaining results in this section are applications of Theorem 5.2 used to derive results about minimum and maximum minimal volume \((n)SSs\). This is done in the same way as when deriving results about minimum and maximum minimal volume \((n)KSSs\) using Theorem 4.2.
Chapter 5. Minimal (n) Separating Systems and Their Volumes

The idea behind the following theorem is that if \( S(n) = S \) then choosing the \( n \) smallest subsets of \([S]\) provides the dual of a minimum volume minimal (n)SS by an application of Theorem 5.2.

**Theorem 5.3.** Assume \( S(n) = S \) and let \( q = \max_{q \in [S]} \left\{ \sum_{i=0}^{q} \binom{s}{i} \leq n \right\} \) and \( t = n - \sum_{i=0}^{q} \binom{s}{i} \). Then \( V_{\text{min}}(S_n) = \sum_{i=0}^{q} i \binom{s}{i} + t(q + 1) \).

**Proof.** The proof is similar to the proof of Theorem 4.2 so its details are not included. \( \square \)

**Corollary 5.4.** If \( S \) is a minimum volume minimal (n)SS with \( S(n) = S \) then there are \( S \) elements of \([n]\) which occur exactly once in \( S \).

**Proof.** By Corollary 5.3, \( S(n) < n \). Then by Theorem 5.3, the dual \( S^* \) of \( S \) must contain at least the empty set and of the singleton subsets of \([S]\). By the definition of the dual, each element of \([S]\) must occur exactly once in \( S \). This proves the corollary. \( \square \)

**Note 5.2.** The converse is not true. For example, \( S = \{12, 13, 14\} \) is a minimal (5)SS in which the elements 2, 3 and 4 occur once in \( S \) but \( S \) is not a minimum volume minimal (5)KSS since a minimum volume minimal (5)SS is \( \{12, 13, 4\} \).

The following theorem is similar to Theorem 5.3 but applies to maximum volume minimal (n)SSs. It shows that choosing the \( n \) largest non-empty subsets of \([S]\) provides the dual of a maximum volume minimal (n)SSs by an application of Theorem 5.2.

**Theorem 5.4.** Assume \( S(n) = S \) and let \( q' = \max_{q' \leq S} \left\{ \sum_{i=0}^{q'} \binom{s}{i} \leq n \right\} \) and \( t' = n - \sum_{i=0}^{q'} \binom{s}{i} \). Then \( V_{\text{max}}(S_n) = \sum_{i=0}^{q'} i \binom{s}{i} + t'(S - q' - 1) \).

**Proof.** The proof is similar to the proof of Theorem 4.3 so its details are not included. \( \square \)
Chapter 5. Minimal \((n)\) Separating Systems and Their Volumes

Corollary 5.5. If \(S\) is a maximum volume minimal \((n)SS\) then there is an element of \([n]\) which occurs in every set in \(S\).

**Proof.** The proof is similar to the proof of Corollary 4.4 so its details are not included. \(\square\)

**Note 5.3.** The converse is not true. For example, in a minimal \((3)KSS \ S = \{12,1\}\) the element 1 occurs in every set in \(S\) but \(S\) is not a maximum volume minimal \((3)KSS\) since a maximum volume minimal \((3)KSS\) is \(\{12,13\}\).

### 5.3 Number of Minimum Volume Minimal \((n)SSs\)

From Theorems 5.3 and 5.4 results about the number of minimum and maximum minimal \((n)SSs\) can be derived.

**Theorem 5.5.** (1) Let \(q, t\) and \(S\) be as in Theorem 5.3. The number of labelled minimum volume minimal \((n)SSs\) is \(N_1 = \binom{s}{q+1} \binom{t}{q+1}^{-1}\).

(2) Let \(q', t'\) and \(S\) be as in Theorem 5.4. The number of labelled maximum volume minimal \((n)SSs\) is \(N'_1 = \binom{s}{q'-1} \binom{t'}{q'-1}^{-1}\).

**Proof.** The proof is similar to the proof of Theorem 4.4 so its details are not included. \(\square\)

**Note 5.4.** If \(t = 0\) (respectively, \(t' = 0\)) then there is a unique minimum (respectively, maximum) volume minimal \((n)SS\).

**Lemma 5.1.** (1) Let \(q, t\) and \(S\) be as in Theorem 5.3. If \(t = 1\) or \(t = \binom{s}{q+1} - 1\) then there is a unique non-isomorphic minimum volume minimal \((n)SS\).

(2) Let \(q', t'\) and \(K\) be as in Theorem 5.4. If \(t' = 1\) or \(t' = \binom{s}{q'-1} - 1\) then there is a unique non-isomorphic maximum volume minimal \((n)SS\).
Chapter 6

Non-Isomorphic Minimal \((n)\) Covering Separating Systems and \((n)\) Separating Systems
Chapter 6. Non-Isomorphic Minimal \((n)KSSs\) and \((n)SSs\)

\subsection{Introduction}

In this chapter the number of non-isomorphic minimal \((n)KSSs\) and \((n)SSs\) are determined for some values of \(n\). Based upon the close relationship between \((n)KSSs\) and \((n)SSs\) the number of non-isomorphic minimal \((n)SSs\) can be derived from the number of non-isomorphic minimal \((n)KSSs\) and vice versa. As mentioned in the introduction, the number of non-isomorphic minimal \((n)KSSs\) is determined first and the number of non-isomorphic minimal \((n)SSs\) is then derived from those results. This is done in Section 6.2 and 6.3 respectively.

The derivation of these values provides a catalogue of non-isomorphic minimal \((n)KSSs\) for \(2 \leq n \leq 7\) and minimal \((n)SSs\) for \(2 \leq n \leq 8\). This catalogue appears in Appendix B.

\subsection{Non-Isomorphic Minimal \((n)KSSs\)}

In this section a \(KSS\) \(\mathcal{K}\) is said to be in \textbf{standard form} if the first set in \(\mathcal{K}\) is the set \([m]\) where \(m\) is the size of the largest set in \(\mathcal{K}\). In each lemma in this section \(\mathcal{K}\) stands for a minimal \((n)KSS\) in standard form for the appropriate value of \(n\). The values stated in each lemma for \(K = K(n)\) and \(V_{\min}(\mathcal{S}_n)\) follow from Corollary 3.3 and Theorem 4.2 respectively. Also note that the catalogue on minimal \((n)KSSs\) which is derived in Lemma 6.2 is complete as an exhaustive search has been made in each case to ensure that all possibilities have been included. This is feasible by using the known constraints on \((n)KSSs\).

The results in this section follow from the results on volumes of \((n)KSSs\) established in Chapter 4. The following theorem is the main result in this section.

\textbf{Theorem 6.1.} For \(K \geq 2\) and \(n = 4, 5, 2^K - 2\) or \(2^K - 1, K(n), d_K(n), V_{\min}(\mathcal{K}_n),\) and \(V_{\max}(\mathcal{K}_n)\) are given in Table 6.1.
Chapter 6. Non-Isomorphic Minimal \((n)KSSs\) and \((n)SSs\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>4</th>
<th>5</th>
<th>(2^K - 2)</th>
<th>(2^K - 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K(n))</td>
<td>3</td>
<td>3</td>
<td>(K)</td>
<td>(K)</td>
</tr>
<tr>
<td>(d_K(n))</td>
<td>10</td>
<td>6</td>
<td>(K)</td>
<td>1</td>
</tr>
<tr>
<td>(V_{\min}(\mathcal{K}_n))</td>
<td>5</td>
<td>7</td>
<td>(K2^{K-1} - K)</td>
<td>(K2^{K-1})</td>
</tr>
<tr>
<td>(V_{\max}(\mathcal{K}_n))</td>
<td>9</td>
<td>10</td>
<td>(K2^{K-1} - 1)</td>
<td>(K2^{K-1})</td>
</tr>
</tbody>
</table>

Table 6.1: Values of \(K(n), d_K(n), V_{\min}(\mathcal{K}_n),\) and \(V_{\max}(\mathcal{K}_n)\)

Theorem 6.1 follows from Corollary 3.2 and Theorems 4.2 and 4.3 and the three results in the remainder of this section, namely, Theorem 6.2 and Lemmas 6.1 and 6.2. Lemma 6.1 follows immediately from Corollaries 4.5 and 3.2.

**Lemma 6.1.** If \(n = 2^K - 1\) then \(K(n) = K\) and \(d_K(n) = 1\).

**Theorem 6.2.** If \(n = 2^K - 2\) then \(K(n) = K\) and \(d_K = K\).

Proof. Assume \(n = 2^K - 2\). Then \(K(n) = K\) by Corollary 3.2. By Theorem 3.5, the dual of any \((n)KSS\) is a collection of \(2^K - 2\) non-empty subsets of \([K]\). Thus each dual contains all but one of the non-empty subsets of \([K]\). For fixed \(i, 1 \leq i \leq K\), there are \(\binom{K}{i}\) choices of a non-empty \(i\)-subset to be excluded from the dual but each of the resultant collections are isomorphic. As \(i\) can take on \(K\) values, there are \(K\) non-isomorphic choices for the excluded set. So there are \(K\) non-isomorphic duals and therefore there are \(K\) exactly non-isomorphic minimal \((n)KSSs\) with \(K(n) = K\).

**Example 6.1.** (1) \(n = 2\): \(K = 2\).

By Theorem 6.2, \(d_K(2) = 2\). The catalogue of non-isomorphic \((2)KSSs\) is:

\[
\mathcal{K}_{2,1} : \begin{array}{c} 1 \\ 2 \end{array} \quad \text{and} \quad \mathcal{K}_{2,2} : \begin{array}{c} 1 \\ 2 \end{array}
\]

(2) \(n = 3\): \(K = 2\).

By Lemma 6.1, \(d_K(3) = 1\). The catalogue of non-isomorphic \((3)KSSs\) is:
Chapter 6. Non-Isomorphic Minimal \((n)KSSs\) and \((n)SSs\)

\[ K_{3,1} : \begin{array}{cc} 1 & 2 \\ 1 & 3 \end{array} \]

Lemma 6.2. Let \(d_K(n)\) be the number of non-isomorphic minimal \((n)KSSs\) with \(K(n) = K\). Then:

1. \(d_K(4) = 10\);
2. \(d_K(5) = 6\).

Proof. The values of \(K(n) = K\) for \(4 \leq n \leq 5\) are given by Corollary 3.2. In the process of determining \(d_K(n)\) all non-isomorphic minimal \((n)KSSs\) for \(4 \leq n \leq 5\) are constructed.

The proof consists of considering the range of possible values of \(m \leq n\) for each \(n\) where \(m\) is the size of the largest set in \(\mathcal{K}\). When minimal \((n)KSSs\) \(\mathcal{K}\) and \(\mathcal{K}'\) with \(\mathcal{K}'\) the complement of \(\mathcal{K}\) are constructed, then \(\mathcal{K}'\) is represented in its equivalent standard form so it may be written as an isomorphic copy of \(\mathcal{K}'\) rather than the complement of \(\mathcal{K}\). Recall that the complement of a \((n)KSS\) \(\mathcal{K}\) is a \((n)KSS\) if and only if no element of \([n]\) occurs in every set in \(\mathcal{K}\).

The derivation of the non-isomorphic \(KSSs\) has been achieved using the underlying structures that a \(KSS\) must have, based upon the values of \(n, K, V_{\text{min}}(\mathcal{K}_n), V_{\text{max}}(\mathcal{K}_n)\) and \(m\). The values of \(V_{\text{min}}(\mathcal{K}_n)\) and \(V_{\text{max}}(\mathcal{K}_n)\) are given by Theorem 4.2 and 4.3 respectively. The underlying structures for fixed \(n\) include the forms of the solutions for smaller \(n\). The recognition of these underlying structures allow a computationally feasible exhaustive construction of all non-isomorphic \(KSSs\) with the given parameters.

1. \(n = 4\): \(K = 3, V(\mathcal{K}) \geq 5\).
   
   (a) \(m \neq 1\) as \(K = 3\) and \(V(\mathcal{K}) \geq 5\).
   
   (b) Assume \(m = 2\). It can be checked by exhaustion that the elements 1 and 2 can be covering separated in no more than three sets including the set \([2]\) as
shown below.

\[
B_1: \begin{array}{cccc}
1 & 2 & 1 & 2 \\
1 & & & \\
\end{array} \\
B_2: \begin{array}{ccc}
1 & & \\
1 & & \\
\end{array} \\
B_3: \begin{array}{ccc}
& & 1 \\
& 2 & \\
\end{array} \\
B_4: \begin{array}{ccc}
& & 1 \\
& & 2 \\
\end{array}
\]

This gives the following three designs when the elements 3 and 4 are included.

Note that \( B_4 \) cannot be used as only one of the elements 3 and 4 can be included given that \( m = 2 \).

\[
\begin{align*}
\mathcal{K}_{4,1} & : 1 & 3 \\
\mathcal{K}_{4,2} & : 1 & 3 \\
\mathcal{K}_{4,3} & : 1 & 3 \\
\end{align*}
\]

Here \( \mathcal{K}_{4,3} \cong \mathcal{K}_{4,3}' \).

(c) Assume \( m = 3 \). It can be assumed that \([3]\) is one of the sets in \( \mathcal{K} \). There are two possible ways to covering separate two of the elements of \([3]\) in the last two sets of \( \mathcal{K} \) as shown in Example 6.1(1) and there is only one way to covering separate the three elements of \([3]\) in the last two sets of \( \mathcal{K} \) as shown in Example 6.1(2). This gives the following six designs when the element 4 is included:

\[
\begin{align*}
\mathcal{K}_{4,4} & : 1 & 4 & 1 & 2 & 3 \\
\mathcal{K}_{4,5} & : 1 & 4 & 2 & 4 \\
\mathcal{K}_{4,6} & : 1 & 2 & 3 \\
\end{align*}
\]

Here \( \mathcal{K}_{4,4} \cong \mathcal{K}_{4,4}' \) and \( \mathcal{K}_{4,5} \cong \mathcal{K}_{4,1}' \).

(d) Assume \( m = 4 \). It can be assumed that \([4]\) is one of the sets in \( \mathcal{K} \). There is only one way to covering separate three of the elements of \([4]\) in the last sets of \( \mathcal{K} \) as shown in Example 6.1(2). It is not possible to covering separate all of the elements of \([4]\) in 2 sets as \( K(4) = 3 \). This gives the following design:
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Therefore there are 10 non-isomorphic minimal \((4)KSSs\). That is, \(d_K(4) = 10\).

(2) \(n = 5\): \(K = 3\), \(V(\mathcal{K}) \geq 7\).

(a) \(m \neq 1\) or \(m \neq 2\) as \(K = 3\) and \(V(\mathcal{K}) \geq 7\).

(b) Assume \(m = 3\). It can be assumed that \([3]\) is one of the sets in \(\mathcal{K}\). The same reasoning as in (c) of (1) above gives the following designs when the elements 4 and 5 are included:

\[
\begin{align*}
\mathcal{K}_{5,1} & : 1 \quad 2 \quad 3 \\
\mathcal{K}_{5,2} & : 1 \quad 4 \quad 5 \\
\mathcal{K}_{5,3} & : 1 \quad 2 \quad 4 \\
\mathcal{K}_{5,4} & : 1 \quad 2 \quad 3
\end{align*}
\]

Here \(\mathcal{K}_{5,2} \cong \mathcal{K}_{5,1}'\).

(c) Assume \(m = 4\). It can be assumed that \([4]\) is one of the sets in \(\mathcal{K}\). The same reasoning as in (d) of (1) above gives the following two designs when the element 5 is included:

\[
\begin{align*}
\mathcal{K}_{5,5} & : 1 \quad 2 \quad 3 \quad 4 \\
\mathcal{K}_{5,6} & : 1 \quad 2 \quad 3 \\
\end{align*}
\]

(d) \(m \neq 5\) as \(K(4) = 3\) so the four elements cannot be covering separated in the two remaining sets.

Therefore there are 6 non-isomorphic minimal \((5)KSSs\). That is, \(d_K(5) = 6\). \(\square\)
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6.3 Non-Isomorphic Minimal \((n)SSs\)

Theorem 6.3. (1) If \(n = 2^p\) then \(d_S(n) = d_K(n-1)\).

(2) If \(n \neq 2^p\) then \(d_S(n) = d_K(n-1) + d_K(n)\).

Proof. This follows from Theorem 3.3.

The next theorem is derived from the results in Chapters 2 and 5 and from the results established in the previous section for the number of non-isomorphic minimal \((n,k)KSSs\). Note that a catalogue of minimal \((n)SSs\) for \(n \leq 8\) can be made using the catalogue of minimal \((n)KSSs\) for \(n \leq 7\) in Section 6.2 and applying Theorem 3.3. This appears in Appendix B.

Theorem 6.4. For \(S \geq 2\) and \(n = 2, 5, 6, 2^S - 1\) or \(2^S\), \(S(n), d_S(n), V_{\min}(S_n),\) and \(V_{\max}(S_n)\) are given in the Table 6.2.

<table>
<thead>
<tr>
<th>(n)</th>
<th>2</th>
<th>5</th>
<th>6</th>
<th>(2^S - 1)</th>
<th>(2^S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S(n))</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>(S)</td>
<td>(S)</td>
</tr>
<tr>
<td>(d_S(n))</td>
<td>1</td>
<td>16</td>
<td>9</td>
<td>(S+1)</td>
<td>1</td>
</tr>
<tr>
<td>(V_{\min}(S_n))</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>(S2^{S-1} - S)</td>
<td>(S2^{S-1})</td>
</tr>
<tr>
<td>(V_{\max}(S_n))</td>
<td>1</td>
<td>10</td>
<td>11</td>
<td>(S2^{S-1})</td>
<td>(S2^{S-1})</td>
</tr>
</tbody>
</table>

Table 6.2: Values of \(S(n), d_S(n), V_{\min}(S_n),\) and \(V_{\max}(S_n)\)

Proof. The values of \(S(n)\) are given by Theorem 2.1. The values of \(V_{\min}(S_n)\) and \(V_{\max}(S_n)\) are given by Theorems 5.3 and 5.4. The values of \(d_S(n)\) follow from Theorems 6.1 and 6.3.
Chapter 7

Minimal \((n, k)\) Covering

Separating Systems
7.1 Introduction

In this chapter \((n, k)\)KSSs are considered. In Section 7.2 several basic results on \((n, k)\)KSSs and minimal \((n, k)\)KSSs are given. Section 7.3 provides lower and upper bounds on \(K(n, k)\) and compares \(K(n, k)\) with \(R(n, k)\). Section 7.3.3 provides some characterisations of minimal \((n, k)\)SSs. In Section 7.4 three cases are given when \(K(n, k)\) achieves the lower bound in Theorem 7.3. This happens when \(n\) is sufficiently large compared to \(k\). The chapter allows the determination of \(K(n, k)\) for all \(n\) with \(k \leq 4\) or \(n \geq \frac{k^2}{2}\).

7.2 \((n)\)KSSs: Basic Results

The section begins with Subsection 7.2.1 which is a collection of results exploring the relationship between \((n, k)\)KSSs and their corresponding \((0, 1)\)-arrays. It also briefly considers relationships between \((n, k)\)KSSs, antichains and \((n, k)\)SSs. Subsection 7.2.2 states a result similar to that of the symmetry lemma for \((n, k)\)CSSs (Lemma 2.2). Subsection 7.2.3 derives values of \(K(n, 1)\) and \(K(n, n-1)\).

7.2.1 \((0, 1)\)-Arrays, Antichains and \((n, k)\)SSs

Let \(M\) be the corresponding \((0, 1)\)-array of a \((n, k)\)KSS \(\mathcal{C}\). In \(M\), let \(C_j\) and \(|C_j|\) denote the \(j\)-th column and the number of 1s in \(C_j\) respectively. The following lemmas concern \(M\).

**Lemma 7.1.** Let \(M\) be the corresponding \((0, 1)\)-array of a \((n, k)\)KSS. Then \(|C_j| \geq 1\) for \(1 \leq j \leq n\).
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Proof. This is a consequence of the fact that each element of \([n]\) must occur in a KSS on \([n]\). 

Lemma 7.2. Let \(\mathcal{K}\) be a \((n,k)\)KSS with the corresponding \((0,1)\)-array \(M\). Then \(\sum_{j=1}^{n} |C_j| = k|\mathcal{K}|\).

Proof. Since \(|A| = k\), for all \(A \in \mathcal{K}\), \(k|\mathcal{K}|\) is the total number of 1s in \(M\). Thus \(\sum_{j=1}^{n} |C_j| = k|\mathcal{K}|\). 

Lemma 7.3 says that a KSS can be represented as a corresponding \((0,1)\)-array in which any two columns are distinct.

Lemma 7.3. Let \(\mathcal{K}\) be a collection of subsets of \([n]\). Let \(M = (m_{ij})\) be a corresponding \((0,1)\)-array of \(\mathcal{K}\). \(\mathcal{K}\) is a covering separating system on \([n]\) if and only if any two columns are distinct.

Proof. Suppose \(\mathcal{K}\) is a KSS on \([n]\) with the corresponding \((0,1)\)-array \(M\). Then \(|C_j| \geq 1\) for each \(j \in [n]\) by Lemma 7.1. Since \(\mathcal{K}\) is a SS for any two distinct elements \(j, l \in [n]\) there exists a set in \(\mathcal{K}\) that separates either \(j\) from \(l\) or \(l\) from \(j\). That is, any two columns in \(M\) are distinct.

Suppose \(M\) is a corresponding \((0,1)\)-array of \(\mathcal{K}\) which satisfies \(|C_j| \geq 1\) for each \(j \in [n]\) and any two columns \(C_j\) and \(C_l\) of \(M\) are distinct. Then there exists \(1 \leq i \leq |\mathcal{K}|\) with \(m_{ij} \neq m_{il}\). Thus only one of \(j\) and \(l\) is in \(A_i \in \mathcal{K}\), so the elements \(j\) and \(l\) are separated from one another by \(\mathcal{K}\). 

Lemma 7.4. Let \(\mathcal{K}\) be a \((n,k)\)KSS with the corresponding \((0,1)\)-array \(M\). If \(|C_i| = |\mathcal{K}|\), then \(|C_i| > |C_j|\) for all \(j \neq i\).

Proof. Suppose \(|C_i| = |\mathcal{K}|\). Assume that there is a \(|C_j|\) such that \(|C_j| = |C_i|\) for \(j \neq l\). This means that \(j\) and \(l\) are in exactly the same set in \(\mathcal{K}\). This contradicts \(\mathcal{K}\) being a KSS.
Lemma 7.5. Let $\mathcal{K}$ be a covering separating system (respectively separating system) on $[n]$ and let $M$ be the corresponding $(0,1)$-array of $\mathcal{K}$.

1. If any two columns of $M$ are interchanged, then the covering separating system (respectively separating system) corresponding to the new $(0,1)$-array is isomorphic to the original one.

2. If any two rows of $M$ are interchanged, then the covering separating system (respectively separating system) corresponding to the new $(0,1)$-array is identical to the original one.

Proof. (1) Let $M^\circ$ be the $(0,1)$-array obtained from $M$ by interchanging any two columns. Interchanging two columns gives a permutation of two elements of $[n]$. Hence the separating system $\mathcal{K}^\circ$ corresponding to $M^\circ$ is isomorphic to $\mathcal{K}$.

(2) This is obvious. □

The next lemma states a simple relationship between $(n,k)KSS$s and $(n,k)SS$s and antichains.

Lemma 7.6. A minimal $(n,k)KSS$ or minimal $(n,k)SS$ is an antichain on $[n]$.

Proof. Note that $n > k$. Let $\mathcal{K}$ be a minimal $(n,k)KSS$ or minimal $(n,k)SS$. Since all members of a minimal $(n,k)KSS$ or minimal $(n,k)SS$ on $[n]$ are distinct and have the same size $k$, $\mathcal{K}$ is an antichain $[n]$. □

Note that Lemma 7.6 refers to a $(n,k)KSS$ $\mathcal{K}$ and not the dual of $\mathcal{K}$ which is an antichain on $|\mathcal{K}|$ by Lemma 2.1.

The following lemma is obvious given the relationship between $SS$s and $KSS$s.

Lemma 7.7. $S(n,k) = \min\{K(n-1,k), K(n,k)\}$.

Note 7.1. It is unknown if $K(n,k)$ is monotonic in $n$ for fixed $k$ for $n \geq 2k$ and so it is not known if $S(n,k) = K(n-1,k)$ for $n \geq 2k$. 

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Chapter 7. Minimal \((n, k)\) Covering Separating Systems

Lemma 7.8. The dual \(\mathcal{K}^*\) of a minimal \((n, k)\)KSS \(\mathcal{K}\) is an equitable completely separating system on \([|\mathcal{K}|]\).

Proof. By Lemma 7.6, a \((n, k)\)KSS \(\mathcal{K}\) is an antichain on \([n]\). By Lemma 2.1, \(\mathcal{K}^*\) the dual of \(\mathcal{K}\) is a completely separating system on \([|\mathcal{K}|]\) in which each element of \([n]\) occurs \(k\) times. Thus \(\mathcal{K}^*\) is an equitable CSS on \([|\mathcal{K}|]\). \(\square\)

7.2.2 A Near Symmetry Lemma

The following lemma is relevant to the near symmetry conjecture (see Chapter 9). It is to be compared with the symmetry lemma for \((n, k)\)CSSs (Lemma 2.2).

Lemma 7.9. Let \(\mathcal{K}\) be a minimal \((n, k)\)KSS in which no element of \([n]\) occurs in every set. Then the complement \(\mathcal{K}'\) of \(\mathcal{K}\) is a \((n, n-k)\)KSS.

Proof. Let \(\mathcal{K}\) be a minimal \((n, k)\)KSS in which no element of \([n]\) occurs in every set. Since there is no element of \([n]\) occurring in every set of \(\mathcal{K}\), each element occurs at least once in \(\mathcal{K}'\). Furthermore, without loss of generality, suppose that for any two distinct elements \(a, b \in [n]\), there exists \(A \in \mathcal{K}\) such that \(a \in A\) and \(b \not\in A\). Hence \(a \not\in A'\) and \(b \in A'\). Thus \(\mathcal{K}'\) is a \((n, n-1)SS\) in which every element occurs at least once. Therefore \(\mathcal{K}'\) is a \((n, n-k)\)KSS. \(\square\)

Note 7.2. \(\mathcal{K}'\) is not always a minimal \((n, n-k)\)KSS. For example, \(\mathcal{K} = \{12, 23, 34, 45\}\) is a minimal \((5, 2)\)KSS but \(\mathcal{K}' = \{123, 125, 145, 345\}\) is not a minimal \((5, 3)\)KSS as \(\{123, 124, 135\}\) is a minimal \((5, 3)\)KSS.

7.2.3 Values of \(K(n, 1)\) and \(K(n, n-1)\)

In this subsection the values of \(K(n, 1)\) and \(K(n, n-1)\) are given.
Lemma 7.10. For all $n > 1$,

$$K(n, 1) = S(n, 1) + 1 = n.$$  

Proof. It is clear that a minimal $(n, 1)KSS$ has exactly $n$ members of size 1. Hence $K(n, 1) = n$. A minimal $(n, 1)SS$ consists of $n - 1$ singleton sets. Hence $S(n, 1) = n - 1$. \hfill \Box

A relationship between $(n, k)KSSs$ and $(n, k)CSSs$ is made by combining Lemma 2.5 and 7.10.

Lemma 7.11. For $n > 1$,

$$K(n, 1) = R(n, 1) = n.$$  

The next lemma considers $K(n, n - 1)$.

Lemma 7.12. Let $\mathcal{K}$ be a minimal $(n, n - 1)KSS$ for $n \geq 3$. Then there exists an element of $[n]$ which occurs in every member of $\mathcal{K}$.

Proof. Let $\mathcal{K}$ be a minimal $(n, n - 1)KSS$. It can be assumed that $A = \{1, 2, \ldots, n - 1\} \in \mathcal{K}$. Since $\mathcal{K}$ is minimal, $\mathcal{K}$ must contain at least enough sets to separate the elements of $A$ from one another. Without loss of generality, these sets must be

$$B_1 = \{1, 2, \ldots, n - 2, n\},$$
$$B_2 = \{1, 2, \ldots, n - 3, n - 1, n\},$$
$$\vdots$$
$$B_{n-3} = \{1, 2, 4, \ldots, n - 1, n\},$$
$$B_{n-2} = \{1, 3, \ldots, n - 1, n\}.$$  

It is clear that $\{B_i, 1 \leq i \leq n - 1\}$ separates each element of $[n - 1]$ from the element $n$ and $A$ separates the element 1 from the element $n$. Thus the element 1 occurs in every set in $\mathcal{K}$. \hfill \Box
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The following lemma follows from Lemma 7.12.

**Lemma 7.13.** For all \(n \geq 3\), \(K(n, n-1) = n - 1\).

### 7.3 Bounds on \(K(n, k)\)

In this section upper and lower bounds on \(K(n, k)\) are derived.

#### 7.3.1 Upper Bounds on \(K(n, k)\)

The next lemma provides an upper bound on \(K(n, k)\).

**Lemma 7.14.** For \(2 \leq k < n\),

\[ K(n, k) < n. \]

**Proof.** For \(k > 1\), \(K(n, k) < n\) follows from consideration of the collection
\[
\{\{1, \ldots, k\}, \{2, \ldots, k+1\}, \ldots, \{n-1, n, 1, \ldots, k-2\}\}.
\]

**Lemma 7.15.** For \(1 \leq k \leq n\), if \(K(n, k) \leq k\) then \(K(n + k + 1, k + 1) \leq k + 1\).

**Proof.** Let \(\mathcal{K} = \{B_1, B_2, \ldots, B_k\}\) be a \((n, k)KSS\) and let \(A = \{n+1, \ldots, n+k+1\}\). Construct \(\mathcal{K}_1\) from \(\mathcal{K}\) and \(A\) as follows.

\[
\mathcal{K}_1 = \{B_1 \cup \{n + 1\}, \ldots, B_k \cup \{n + k\}, A\}.
\]

\(\mathcal{K}_1\) is a \((n + k + 1, k + 1)KSS\) since the elements of \([n]\) are separated from one another by \(B_i \cup \{n + i\}, 1 \leq i \leq k\). The elements of \(A\) are separated from each element of \([n]\) by \(A\) and from one another by \(B_i \cup \{n + i\}, 1 \leq i \leq k\) and \(A\). Therefore \(K(n + k + 1, k + 1) \leq k + 1\).

The following theorem provides an interesting relationship between the size of a minimal \((n, k)KSS\) and a corresponding minimal \((n, k)CSS\).
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Theorem 7.1. For \(2 \leq k < n\),

\[ K(n, k) \leq R(n, k) - 1. \]

Proof. Let \(C\) be a minimal \((n, k)\)CSS and for some \(A \in C\) let \(K = C \setminus A\). Since each element of \([n]\) occurs at least twice in \(C\), when \(A\) is removed from \(C\), each element of \([n]\) occurs in \(K\). If there are elements \(a, b \in [n]\) which are not separated in \(K\), the elements \(a\) and \(b\) cannot be completely separated in \(C\) as at least two additional sets are required to completely separate them from one another. Therefore \(K\) is a \((n, k)\)KSS. \(\square\)

7.3.2 A Lower Bound on \(K(n, k)\)

In this subsection a lower bound for \(K(n, k)\) is given. This result is a consequence of Theorem 7.2 below. Theorem 7.2 and its two corollaries are the \((n, k)\)KSS counterparts of Theorem 4.1, Corollaries 4.1 and 4.2 respectively. Their proofs are omitted as they are minor variations on the proofs for the \((n)\)KSS case.

Theorem 7.2. In any minimal \((n, k)\)KSS, there are at most \(K(n, k)\) elements which occur exactly once.

Corollary 7.1. In any minimal \((n, k)\)KSS, there are at least \(n - K(n, k)\) elements which occur at least twice.

Corollary 7.2. Let \(K\) be a minimal \((n, k)\)KSS. Then \(V(K) \geq 2n - |K|\).

The next theorem provides a lower bound on \(K(n, k)\) for all \(n\) and \(k\). It can be compared with Lemma 2.4 which says that \(R(n, k) \geq \left\lceil \frac{2n}{k} \right\rceil\) for each \(n\) and \(k\).

Theorem 7.3. For \(1 \leq k < n\),

\[ K(n, k) \geq \left\lceil \frac{2n}{k + 1} \right\rceil. \]
Proof. Let $\mathcal{K}$ be any minimal $(n, k)KSS$ with $K(n, k) = K$. Then, by Corollary 7.2, $V(\mathcal{K}) \geq 2n - K$. Also $V(\mathcal{K}) = kK$ as $\mathcal{K}$ contains $K$ sets of size $k$. Thus $kK \geq 2n - K$. By rearranging the inequality, $K(k + 1) \geq 2n$ and so $K(n, k) \geq \left\lceil \frac{2n}{k+1} \right\rceil$ as $K(n, k)$ is an integer. 

7.3.3 Characterisations of $(n, k)KSS$s

The results in this section are due to Roberts [21]. The next lemma, Lemma 7.16, is the $(n, k)KSS$ counterpart of Lemma 3.8. The subsequent results arise from the characterisation of $(n, k)KSS$s in Lemma 7.16. It will be seen in Chapter 8 that these results help to determine some values of $K(n, k)$ for minimal $(n, k)KSS$s which do not achieve the lower bound in Theorem 7.3.

Lemma 7.16. Let $\mathcal{K}$ be a $(n, k)KSS$ with $|\mathcal{K}| = K$. Then:

(i) there is at most one 1-element in each set in $\mathcal{K}$;

(ii) at most $K - 1$ 2-elements in each set in $\mathcal{K}$;

(iii) there are at most $\frac{K(K-1)}{2}$ 2-elements in $\mathcal{K}$.

Proof. The proof is left to the reader as it is a minor variation on the proof of Lemma 3.8. 

Lemma 7.16 provides an upper bound on the number of 2-elements in a minimal $(n, k)KSS$. The next lemma provides a lower bound on the number of 2-elements in a minimal $(n, k)KSS$.

Lemma 7.17. Let $\mathcal{K}$ be a minimal $(n, k)KSS$ with $|\mathcal{K}| = K$. Let $r$ be the minimum number of 2-elements in $\mathcal{K}$. Then $r \geq \max\{0, 3n - K(k + 2)\}$.

Proof. The lemma is obvious if $3n \leq K(k + 2)$. Assume $3n > K(k + 2)$. Note that as the volume of $\mathcal{K}$ is fixed, the possible number of elements in $\mathcal{K}$ with cardinality greater than 2 is maximised when $\mathcal{K}$ contains the maximum possible
number of 1-elements. By Theorem 7.2 $\mathcal{K}$ can contain at most $K$ 1-elements and so there are at least $n - K$ elements which must occur at least twice in $\mathcal{K}$. These elements must fill $V(\mathcal{K}) - K = Kk - K = K(k - 1)$ places in $\mathcal{K}$.

Set $u = K(k - 1) - 2(n - K) = K(k + 1) - 2n$. It can be seen that $u$ is the maximum number of elements which can occur more than twice in $\mathcal{K}$ (by the pigeon-hole principle). Thus $n - K - u = 3n - K(k + 2)$ is the minimum number of 2-elements which must occur in $\mathcal{K}$.

Note 7.3. Lemma 7.17 is part of the motivation for the development of the Method $M_3$ for constructing minimal $(n, k)KSS$s in Section 8.2.

The restriction on the size of the sets in $(n, k)KSS$s and Lemma 3.8 allows the derivation of the next three results.

**Lemma 7.18.** Let $\mathcal{K}$ be a $(n, k)KSS$ with $|\mathcal{K}| = K$. Then there are at least $n - \frac{K(K+3)}{2}$ elements of $\mathcal{K}$ which occur more than twice in $\mathcal{K}$.

*Proof.* By Lemma 7.16 $\mathcal{K}$ contains at most $K$ 1-elements and at most $\frac{K(K-1)}{2}$ 2-elements. The lemma follows from this. $\square$

**Theorem 7.4.** Let $\mathcal{K}$ be a $(n, k)KSS$ with $|\mathcal{K}| = K$. Then $V(\mathcal{K}) \geq 3n - \frac{K(K+3)}{2}$.

*Proof.* The volume of $\mathcal{K}$ has to be sufficiently large to include all occurrences of the 1-elements, the 2-elements and the elements which occur more than twice in $\mathcal{K}$. Thus, by Lemmas 7.16 and 7.18, $V(\mathcal{K}) \geq K + 2\frac{K(K-1)}{2} + 3(n - \frac{K(K+3)}{2}) = 3n - \frac{K(K+3)}{2}$. $\square$

**Corollary 7.3.** Let $\mathcal{K}$ be a $(n, k)KSS$ with $|\mathcal{K}| = K$. Then $kK \geq 3n - \frac{K(K+3)}{2}$.

*Proof.* The corollary follows from Theorem 7.4 as the volume of a $(n, k)KSS$ of size $K$ is $kK$. $\square$
This corollary is very useful in increasing the lower bound on $K(n, k)$ beyond the bound in Theorem 7.3 for many values of $n$ and $k$.

Example 7.1. By Theorem 7.3 $K(12, 5) \geq 4$. Corollary 7.3 improves this bound to $K(12, 5) \geq 5$.

7.4 Cases when $K(n, k)$ Achieves the Lower Bound

The main results in this section, Theorems 7.5, 7.6 and 7.7, are stated first. The remaining of the section is devoted to proving the theorems. The main theorems show that the lower bound on $K(n, k)$ in Theorem 7.3 is attained for sufficiently large $n$ compared to $k$. Theorem 7.5 can be compared with Theorem 2.6 which says that $R(n, k) = \lceil \frac{2n}{k} \rceil$ for $n \geq \binom{k+1}{2}$, $1 \leq k < n$.

Theorem 7.5. For $n \geq \binom{k+1}{2}$, $1 \leq k < n$,

$$K(n, k) = \left\lceil \frac{2n}{k+1} \right\rceil \geq k.$$ 

Theorem 7.6. For all integers $k \geq 3$, if $\frac{k^2}{2} \leq n \leq \binom{k+1}{2} - 1$,

$$K(n, k) = \left\lceil \frac{2n}{k+1} \right\rceil = k.$$

Theorem 7.7. For $\binom{k}{2} \leq n \leq \binom{k}{2} + 1$ and $k \geq 5$,

$$K(n, k) = \left\lceil \frac{2n}{k+1} \right\rceil = k - 1.$$ 

Note 7.4. Theorem 2.6 shows that $R(n, k)$ is not monotonic in $n$ for fixed $k$ with $\binom{k+1}{2} - 2 \leq n \leq \binom{k+1}{2}$ as for $n = \binom{k+1}{2}$, $R(n - 2, k) < R(n - 1, k)$ and $R(n, k) < R(n - 1, k)$. Theorems 7.5 and 7.6 show that $K(n, k)$ is monotonic in $n$ for fixed $k$ and the same values of $n$. It is unknown if $K(n, k)$ is always monotonic in $n$ for fixed $k$. 

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7.4.1 Proof of Theorem 7.5

The proof of Theorem 7.5 for \(k \geq 3\) is split into three parts considered as Theorems 7.8, 7.9 and 7.10. The case when \(k = 1\) follows from Lemma 7.10.

**Theorem 7.8.** For \(n \geq 3\),

\[
K(n, 2) = \left\lceil \frac{2n}{k+1} \right\rceil \geq 2.
\]

**Proof.** Assume \(2 \leq p < n\) and \(3(p - 1) < 2n \leq 3p\) for all \(p, n\). Then \(\frac{3(p-1)}{k+1} < \frac{2n}{k+1} \leq \frac{3p}{k+1}\), which is \(p - 1 < \frac{2n}{3} \leq p\) when \(k = 2\). This gives \(\left\lceil \frac{2n}{3} \right\rceil = p\). The proof is completed by showing that \(K(n, 2) = p\). Let \(C_1 = \{\{i\}, 1 \leq i \leq p\}\) and let \(M_1\) be the \(p \times 1\) array with the element in the row \(i\) being \(i\). By Lemma 2.6, \(R(n-p, 1) = n-p\). Let \(C_2 = \{\{p+i\}, 1 \leq i \leq n-p\}\) be a minimal \((n-p, 1)\)CSS on \([n] \setminus [p]\) and let \(M_2\) be the \((n-p) \times 1\) array with the element in the row \(i\) being \(p+i\). Let \([M_1, M_2]\) be the augmented array of \(M_1\) and \(M_2\). Now construct a \(p \times 2\) array \(M\) from \([M_1, M_2]\) by duplicating each row of \(M_2\) at least twice to fill the spaces below the block \(M_2\) in \([M_1, M_2]\). By Lemma 2.6, \(R(n-p, 1) = n-p \leq p/2\), \([M_1, M_2]\) has enough spaces to satisfy this condition.

Let \(K = \{K_i : K_i\) is the row \(i\) of \(M\}\).

It is now shown that \(K\) is a \((n, 2)\)KSS. Each element of \([p]\) occurs in distinct set in \(K\) so they are separated from one another. By the construction of \(K\), each element of \(\{p+1, p+2, \ldots, n\}\) occurs in distinct set and in at least two sets in \(K\), hence they are separated from one another and from each element of \([p]\). Thus \(K\) is a \((n, 2)\)KSS and \(K(n, 2) \leq p\). By Theorem 7.3, \(K(n, 2) \geq \left\lceil \frac{2n}{3} \right\rceil = p\). Therefore \(K(n, 2) = p\) as required. \(\square\)

**Theorem 7.9.** If \(n = \binom{k+1}{2}\), \(2 < k < n\), then

\[
K(n, k) = \left\lceil \frac{2n}{k+1} \right\rceil = k.
\]
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Proof. Let \(C_1 = \{\{i\}, 1 \leq i \leq k\}\) and let \(M_1\) be the \(k \times 1\) array with the element in the row \(i\) being \(i\). Let \(C_2 = \{B_1, B_2, ..., B_{R(n-k,k-1)}\}\) be a minimal \((n-k, k-1)CSS\) on \([n]\setminus[p]\) and let \(M_2\) be the \(R(n-k, k-1) \times (k-1)\) array with the element in the row \(i\) being the elements of \(B_i\), \(1 \leq i \leq R(n-k, k-1)\).

By Lemma 2.7, \(R(n-k, k-1) = k\) so that \(M_1\) and \(M_2\) both contain \(k\) rows. Let \(M\) be the \(k \times k\) augmented array \([M_1, M_2]\).

Let \(\mathcal{K} = \{K_i : K_i\text{ is the row }i\text{ of }M\}\). It is now shown that \(\mathcal{K}\) is a \((n,k)KSS\). Each element of \([k]\) occurs in distinct set in \(\mathcal{K}\) so they are separated from one another and separated from each element of \([n]\setminus[k]\) as each element of \([n]\setminus[k]\) occurs at least twice in \(M\). Of course, the elements of \([n]\setminus[k]\) are separated from one another as \(M_2\) is the array representing of a CSS. Thus \(\mathcal{K}\) is a \((n,k)KSS\) and \(K(n,k) \leq k\). By Theorem 7.3, \(K(n,k) \geq \left\lfloor \frac{(k+1)k}{k+1} \right\rfloor = k\). Therefore \(K(n,k) = k\) as required.

Note 7.5. In the proof of Theorem 7.9 the minimal \((n-k, k-1)CSS\) \(C_2\) is unique. In Section 8.2 a method, called Construction \(M\), to build such a CSS is given.

Theorem 7.10. For \(n > \left(\frac{k+1}{2}\right)\) and \(k > 2\),

\[
K(n,k) = \left\lceil \frac{2n}{k+1} \right\rceil \geq k.
\]

Proof. Assume \(2 < k < p < n\) and \((k+1)(p-1) < 2n \leq (k+1)p\) for all \(k, p\) and \(n\). Then \(n > \frac{(k+1)(p-1)}{2} \geq \frac{(k+1)k}{k+1} = \left(\frac{k+1}{2}\right)\) and \(\frac{(k+1)(p-1)}{k+1} < \frac{2n}{k+1} \leq \frac{(k+1)p}{k+1}\) so that \((p-1) < \frac{2n}{k+1} \leq p\) and thus \(\left\lceil \frac{2n}{k+1} \right\rceil = p\). The proof is completed by showing that \(K(n,k) = p\).

Let \(C_1 = \{\{i\}, 1 \leq i \leq p\}\) and let \(M_1\) be the \(p \times 1\) array with the element in the row \(i\) being \(i\). Let \(C_2 = \{B_1, B_2, ..., B_{R(n-p,k-1)}\}\) be a minimal \((n-p, k-1)CSS\) and let \(M_2\) be the \(R(n-p, k-1) \times (k-1)\) array with the element in the row \(i\) being the elements of \(B_i\), \(1 \leq i \leq R(n-p, k-1)\). Let \([M_1, M_2]\) be the augmented array of \(M_1\) and \(M_2\). Now construct a \(p \times k\) array \(M\) from \([M_1, M_2]\) by duplicating
some rows of $M_2$ to fill the spaces below the block $M_2$ in $[M_1, M_2]$ if there are. By Lemma 2.8, $R(n - p, k - 1) \leq p$, so there are two possible cases:

(i) If $R(n - p, k - 1) < p$, then there are some spaces below the block $M_2$ in $[M_1, M_2]$. To form the array $M$ it needs to duplicate some rows of $M_2$ to fill these spaces. Note that any row can be chosen equally and as many as enough to obtain $M$.

(ii) If $R(n - p, k - 1) = p$, then $M_1$ and $M_2$ both contain $p$ rows. Hence $M = [M_1, M_2]$.

Let $C = \{ K_i : K_i$ is the row $i$ of $M \}$. It is now shown that $C$ is a $(n, k)KSS$. Each element of $[p]$ occurs in distinct set in $C$ so they are separated from one another and separated from each element of $[n] \setminus [p]$ as each element of $[n] \setminus [p]$ occurs at least twice in $M$. Of course, the elements of $[n] \setminus [p]$ are separated from one another as $M_2$ is the array representing of a CSS. Thus $C$ is a $(n, k)KSS$ and $K(n, k) \leq p$. By Theorem 7.3, $K(n, k) \geq \lceil \frac{2n}{k+1} \rceil = \frac{(k+1)p}{k+1} = p$. Therefore $K(n, k) = p$ as required. \qed

Note 7.6. Theorem 7.10 means that $K(n, k)$ is known for each $k$ except for a finite number of values in each case.

### 7.4.2 Proof of Theorem 7.6

The proof of Theorem 7.6 requires the following four lemmas. Lemma 7.19 is easy to prove and the proof is left to the reader.

**Lemma 7.19.** For all integers $k \geq 3$,

1. There are $\left\lfloor \frac{k^2}{2} \right\rfloor$ integers $n$ which satisfy $\frac{k^2}{2} \leq n \leq \left( \frac{k+1}{2} \right) - 1$.
2. Let $N_1$ be the number of integers $n$ which satisfy $\frac{k^2}{2} \leq n \leq \left( \frac{k+1}{2} \right) - 1$. Let $N_2$ be the number of integers $n'$ which satisfy $\frac{(k+1)^2}{2} \leq n' \leq \left( \frac{k+2}{2} \right) - 1$. Then:
   a) If $k$ is odd, then $N_2 - N_1 = 1$. 

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b) If \(k\) is even, then \(N_1 = N_2\).

Lemma 7.20. Let \(n_i, 1 \leq i \leq \frac{k-1}{2}\), be an integer which satisfies \(\frac{k^2}{2} \leq n_i \leq (\frac{k+1}{2}) - 1\) with \(n_i \leq n_j\) if \(i \leq j\). Let \(n_i', 1 \leq i \leq \frac{k+1}{2}\), be an integer which satisfies \(\frac{(k+1)^2}{2} \leq n_i' \leq (\frac{k+2}{2}) - 1\) with \(n_i' \leq n_j'\) if \(i \leq j\). If \(k \geq 3\) is odd then \(n_1' - n_1 = k\) and \(n'_{i+1} - n_i = k + 1\) for \(1 \leq i \leq \frac{k-1}{2}\).

Proof. Let \(n_i = \frac{k^2}{2} + (i - 1)\) for \(1 \leq i \leq \frac{k-1}{2}\) and \(n_i' = \frac{(k+1)^2}{2} + (i - 1)\) for \(1 \leq i \leq \frac{k+1}{2}\). Then \(n_i' - n_1 = \frac{(k+1)^2}{2} - \frac{k^2}{2} = k\) and \(n_{i+1}' - n_i = \frac{(k+1)^2}{2} + i - \frac{k^2}{2} - (i - 1) = k + 1\) for \(1 \leq i \leq \frac{k-1}{2}\) as required.

Lemma 7.21. Let \(n_i, 1 \leq i \leq \frac{k}{2}\), be an integer which satisfies \(\frac{k^2}{2} \leq n_i \leq (\frac{k+1}{2}) - 1\) with \(n_i \leq n_j\) if \(i \leq j\). Let \(n_i', 1 \leq i \leq \frac{k}{2}\), be an integer which satisfies \(\frac{(k+1)^2}{2} \leq n_i' \leq (\frac{k+2}{2}) - 1\) with \(n_i' \leq n_j'\) if \(i \leq j\). If \(k \geq 3\) is even then \(n_i' - n_i = k + 1\) for \(1 \leq i \leq \frac{k}{2}\).

Proof. Let \(n_i = \frac{k^2}{2} + (i - 1)\) and \(n_i' = \frac{(k+1)^2}{2} + (i - 1)\) for \(1 \leq i \leq \frac{k}{2}\). Then \(n_i' - n_i = \frac{(k+1)^2 + 1}{2} - \frac{k^2}{2} = k + 1\) as required.

Lemma 7.22. \(K(5, 3) = 3\).

Proof. By Theorem 7.3, \(K(5, 3) \geq 3\). As \(\{123, 124, 135\}\) is a \((5, 3)KSS\), \(K(5, 3) = 3\).

Note 7.7. By Lemmas 7.13 and 7.22, and Theorem 7.5, \(K(n, 3)\) is known for all \(n\).

Proof of Theorem 7.6

Proof. The proof is by induction on \(k\) with the base case being \(K(5, 3) = 3\) with \(\{123, 124, 135\}\) being a minimal \((5, 3)KSS\). Note that Lemma 7.19 says that the number of integers \(n'\) is greater than the number of integer \(n\) by one if \(k\) is odd. Otherwise they are equal. There are two cases to consider.
Case 1: $k > 3$ is odd.

Let $n_i$ and $n'_i$ be as in Lemma 7.20. Then there are two subcases:

(i) Assume $K(n_i, k) = k$. Let $\mathcal{K}$ be a minimal $(n_i, k)KSS$ represented as a $k \times k$ array $M$. A $(n'_i, k+1)KSS$ $\mathcal{K}'$ can be constructed from $\mathcal{K}$ by appending an extra row and column to $M$ with the $j$-th element of the new row and column being $n_1 + j$ for $1 \leq j \leq k$ and with $m_{k+1,k+1} = 1$. This process assures the separation property on $[n'_i]$ because: the separation property on $[n_i]$ is still preserved; the elements of $\{n_i + j : 1 \leq j \leq k\}$ are separated from one another and from the element 1 by the first $k$ rows of $\mathcal{K}'$; and the elements of $\{n_i + j : 1 \leq j \leq k\}$ are separated from the elements of $[n_i] \setminus \{1\}$ by the last row of $\mathcal{K}'$. Hence $\mathcal{K}'$ is a $(n'_i, k+1)KSS$. Thus, by Theorem 7.3, $\mathcal{K}'$ is a minimal $(n'_i, k+1)KSS$. Therefore $K(n'_i, k+1) = k + 1$.

(ii) For $1 \leq i \leq \frac{k-1}{2}$, assume $K(n_i, k) = k$. Let $\mathcal{K}$ be a minimal $(n_i, k)KSS$ represented as a $k \times k$ array $M$. A $(n'_{i+1}, k+1)KSS$ $\mathcal{K}'$ can be constructed from $\mathcal{K}$ by appending an extra row and column to $M$ with the $j$-th element of the new row and column being $n_i + j$ for $1 \leq j \leq k + 1$. This process assures the separation property on $[n'_i]$ because the separation property on $[n_i]$ is still preserved and the elements of $\{n_i + j : 1 \leq j \leq k + 1\}$ are separated from one another by the rows of $\mathcal{K}'$ and from the elements $[n_i]$ by the last row of $\mathcal{K}'$. Hence $\mathcal{K}'$ is a $(n'_{i+1}, k+1)KSS$. Thus, by Theorem 7.3, $\mathcal{K}'$ is a minimal $(n'_{i+1}, k+1)$. Therefore $K(n'_{i+1}, k+1) = k + 1$.

Case 2: $k > 3$ is even.

Let $n_i$ and $n'_i$ be as in Lemma 7.21. For $1 \leq i \leq \frac{k}{2}$, assume $K(n_i, k) = k$. Let $\mathcal{K}$ be a minimal $(n_i, k)KSS$ represented as a $k \times k$ array $M$. A $(n'_i, k+1)KSS$ $\mathcal{K}'$ can be constructed from $\mathcal{K}$ by appending an extra row and column to $M$ with the $j$-th element of the new row and column being $n_i + j$ for $1 \leq j \leq k + 1$. This process assures the separation property on $[n'_i]$ because the separation property on $[n_i]$ is still preserved and the elements of $\{n_i + j : 1 \leq j \leq k + 1\}$ are separated
from one another by the rows of $\mathcal{K}'$ and from the elements of $[n_i]$ by the last row of $\mathcal{K}'$. Hence $\mathcal{K}'$ is a $(n'_i, k + 1)KSS$. Thus, by Theorem 7.3, $\mathcal{K}'$ is a minimal $(n'_i, k + 1)KSS$. Therefore $K(n'_i, k + 1) = k + 1$.
This proves the theorem.

See Appendix C for an example of the construction in Theorem 7.6.

### 7.4.3 Proof of Theorem 7.7

The following lemma is needed to prove Theorem 7.7.

**Lemma 7.23.** $K(6, 4) = 4.$

**Proof.** Assume that $\mathcal{K}$ is a minimal $(6, 4)KSS$ with $|\mathcal{K}| = 3$. It can be assumed that $A = \{1, 2, 3, 4\} \in \mathcal{K}$. By Theorem 6.4, there is a unique way of separating the elements of $A$ in 2 sets namely using the sets $\{1, 2\}$ and $\{1, 3\}$. It is not possible to append the elements 5 and 6 to these two sets to form 4-sets and simultaneously separate 5 and 6 from one another. Thus $K(6, 4) \neq 3$. To see that $K(6, 4) = 4$ consider the $(6, 4)KSS$ $\mathcal{K} = \{1234, 1235, 2456, 3456\}$. 

**Proof of Theorem 7.7**

**Proof.** Assume that $n = \binom{k}{2}$ and $k \geq 5$. The proof is by induction on $k$ with the base case being $K(10, 5) = 4$. To prove this base case let $\mathcal{K}$ be a minimal $(6, 4)KSS$ on $[6]$ defined in Lemma 7.23 and represented as a $4 \times 4$ array $M$. A $(10, 5)KSS$ $\mathcal{K}_1$ can be constructed from $\mathcal{K}$ by appending an extra column to $M$ with the $j$-th element of the new column being $6 + j, 1 \leq j \leq 4$. It is clear that each element of $[10]$ is separated from one another by the rows of the obtained $4 \times 5$ array. Hence $\mathcal{K}_1$ is a $(10, 5)KSS$. By Theorem 7.3, $\mathcal{K}_1$ is a minimal $(10, 5)KSS$. Therefore $K(10, 5) = 4$. 

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Assume that \(K(n, k) = k - 1\) for \(k > 5\). It will be proved that \(K(n', k') = k' - 1\) where \(n' = \binom{k+1}{2}\) and \(k' = k + 1\). Let \(\mathcal{K}\) be a \((n, k)\)KSS represented as a \((k-1) \times k\) array \(M\). A \((n', k')\)KSS \(\mathcal{K}'\) can be constructed from \(\mathcal{K}\) by appending an extra row and column to \(M\) to form \(M'\) with the \(j\)-th element of the new row and column being \(n + j, 1 \leq j \leq k\) and the element \(m'_{k,k+1}\) being any element of \([n]\) which has occurred at least twice in \(\mathcal{K}\). Here the element 1 is chosen. This process assures the separation property on \([n']\) because: the separation property on \([n]\) is still preserved; the elements of \(\{n + j : 1 \leq j \leq k\}\) are separated from one another by the first \(k\) rows of \(\mathcal{K}'\); and the elements of \([n]\) are separated from the elements of \(\{n + j : 1 \leq j \leq k\}\) by the first \(k\) rows of \(\mathcal{K}'\). Hence \(\mathcal{K}'\) is a \((n', k')\)KSS. Thus, by Theorem 7.3, \(\mathcal{K}'\) is a minimal \((n', k')\)KSS. Therefore \(K(n', k') = k' - 1\).

Assume that \(n = \binom{k}{2} + 1\) and \(k \geq 5\). The proof is by induction on \(k\). By Corollary 4.6, \(K(7, 4) = 3\). This is the base case. Assume \(K(n, k) = k - 1\) for \(k > 4\). It will be proved that \(K(n', k') = k' - 1\) where \(n' = \binom{k+1}{2} + 1\) and \(k' = k + 1\).

Let \(\mathcal{K}\) be a minimal \((n, k)\)KSS represented as a \((k-1) \times k\) array \(M\). A \((n', k')\)KSS \(\mathcal{K}'\) can be constructed from \(\mathcal{K}\) by appending an extra row and column to \(M\) to form \(M'\) with the \(j\)-th element of the new row and column being \(n + j, 1 \leq j \leq k\) and the element \(m'_{k,k+1}\) being any element of \([n]\) which has occurred at least twice in \(\mathcal{K}\). Here the element 2 is chosen rather than the element 1. This is for later convenience when proving Lemma 9.1. This process assures the separation property on \([n']\) because the separation property on \([n]\) is still preserved, the elements of \(\{n + j : 1 \leq j \leq k\}\) are separated from one another by the first \(k\) rows of \(\mathcal{K}'\) and the elements of \([n]\) are separated from the elements of \(\{n + j : 1 \leq j \leq k\}\) by the first \(k\) rows of \(\mathcal{K}'\). Hence \(\mathcal{K}'\) is a \((n', k')\)KSS. Thus, by Theorem 7.3, \(\mathcal{K}'\) is a minimal \((n', k')\)KSS. Therefore \(K(n', k') = k' - 1\).

See Appendix D for an example of the construction in Theorem 7.7.

Note 7.8. By Lemma 7.13, Theorems 7.5, 7.6 and 7.7, \(K(n, 4)\) is known for all \(n\).
Chapter 8

Particular Values of $K(n, k)$
Chapter 8. Particular Values of $K(n, k)$

8.1 Introduction

In Chapter 7 the values of $K(n, k)$ for $k \leq 4$ or $n \geq \frac{k^2}{2}$ were given. This chapter determines the values of $K(n, k)$ for $5 \leq k \leq 9$ and $2k \leq n < \frac{k^2}{2}$. Thus, by the results in Chapter 7, $K(n, k)$ will be known for all $n \geq 2k$ with $k \leq 9$. Some values of $K(n, k)$ for $10 \leq k \leq 15$ and $2k \leq n \leq 40$ are also determined here. The cases included here are sufficient to fully illustrate the three methods for constructing minimal $(n, k)KSS$s given in Section 8.2. The remaining values of $K(n, k)$ for $10 \leq k \leq 15$ and $2k \leq n \leq 40$ are determined in Appendix E. The values of $K(n, k)$ for $k \leq 15$ and $n < 2k$ are determined in Chapter 9.

The underlying method used to find particular values of $K(n, k)$ is to find a lower bound on $K(n, k)$ and then to use known minimal $(n_1, k_1)KSS$s and $(n_2, k_2)KSS$s to construct a minimal $(n, k)KSS$ where either $n = n_1 + n_2$ and $k = k_1 + k_2$ or $n - 1 = n_1 + n_2$ and $k = k_1 + k_2$. There are three variations on this general approach. These are described in detail in the following section.

The minimal $(n_1, k_1)KSS$s and $(n_2, k_2)KSS$s used will come from the proofs of previous theorems or lemmas. Many of them also appear in Appendices C, D, F and G.

Note that the $k$-sets in any minimal $(n, k)KSS$ with $K(n, k) = K$ are represented as the rows of a $K \times k$ array. The highlighted elements are those which are duplicated to fill the spaces in the augmented arrays used for two of the methods. Extra spaces left in some rows are to highlight the structures of the minimal $(n_1, k_1)KSS$ and $(n_2, k_2)KSS$ used to form the augmented array.
8.2 Methods for Constructing Minimal $(n, k)KSS$s

Three methods are described to show how a minimal $(n, k)KSS$ can be formed from known minimal $(n_1, k_1)KSS$s and $(n_2, k_2)KSS$s. The choice of method depends on $n, k, n_1, n_2, k_1$ and $k_2$.

**Method $M_1$**

Assume that $n_1 = k, n = n_1 + n_2$ and $k = k_1 + k_2$. To form a minimal $(n, k)KSS$ $\mathcal{K}$ on $[n]$ append a minimal $(n_1, k_1)KSS$ $\mathcal{K}_1$ on $[n_1] = [k]$ to a minimal $(n_2, k_2)KSS$ $\mathcal{K}_2$ on $[n] \setminus [k]$ with $K(n_1, k_1) = K_1 \leq |\mathcal{K}| - 1$ or $K(n_2, k_2) = K_2 \leq |\mathcal{K}| - 1$ and use $[k]$ as the first row of the array representing $\mathcal{K}$. Here either $K_1$ or $K_2$ but not both can be strictly less than $|\mathcal{K}| - 1$. If one of $K_1$ and $K_2$ is strictly less than $|\mathcal{K}| - 1$ then some rows of $\mathcal{K}_1$ or $\mathcal{K}_2$ are duplicated to fill the spaces under it.

(End of Method $M_1$)

It is clear that the elements of $[n]$ are covering separated from one another since: the elements of $[k]$ are covering separated from one another by a minimal $(k, k_1)KSS$ on $[k]$ and from the elements of $[n] \setminus [k]$ by the first set $[k]$; the elements of $[n] \setminus [k]$ are covering separated from one another by a minimal $(n_2, k_2)KSS$ on $[n] \setminus [k]$.

**Example 8.1.** (1) A minimal $(9, 5)KSS$ on $[9]$ can be constructed from a minimal $(5, 3)KSS$ on $[5]$ (see Lemma 7.22) and a minimal $(4, 2)KSS$ on $[9] \setminus [5]$ (see Theorem 7.8) as follows.
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1 2 3 4 5
1 2 3 6 9
1 2 4 7 9
1 3 5 8 9

(2) A minimal $(13, 6)KSS$ on $[13]$ can be constructed from a minimal $(6, 3)KSS$ on $[6]$ (see Theorem 7.9) and a minimal $(7, 3)KSS$ on $[13] \setminus [6]$ (see Theorem 7.10) as follows, with the last row of the $(6, 3)KSS$ being duplicated.

1 2 3 4 5 6
1 4 5 7 11 12
2 4 6 8 11 13
3 5 6 9 12 13
3 5 6 10 12 13

Method $M_2$

Assume that $n_1 = k - 1$, $n = n_1 + n_2 + 1$ and $k = k_1 + k_2$. To form a minimal $(n, k)KSS$ $K$ append a minimal $(n_1, k_1)KSS$ on $[n_1] = [k - 1]$ to a minimal $(n_2 - 1, k_2)KSS$ on $[n] \setminus [k]$ with $K(n_1, k_1) = K_1 \leq |K| - 1$ or $K(n_2, k_2) = K_2 \leq |K| - 1$ and use $[k]$ as the first row of the array representing $K$. Here either $K_1$ or $K_2$ but not both can be strictly less than $|K| - 1$. If one of $K_1$ and $K_2$ is strictly less than $|K| - 1$ then some rows of $K_1$ or $K_2$ are duplicated to fill the spaces under it.

(End of Method $M_2$)

It is clear that the elements of $[n]$ are covering separated from one another since: the elements of $[k]$ are separated from one another by a minimal $(k - 1, k_1)KSS$ on $[k - 1]$ and the first set $[k]$; the elements of $[n] \setminus [k]$ are separated from one another and from the elements of $[k]$ by a minimal $(n_2, k_2)KSS$ $[n] \setminus [k]$. 
Example 8.2. A minimal $(12, 5)KSS$ on $[12]$ can be constructed from the minimal $(4, 1)KSS$ (see Lemma 7.10) and a minimal $(7, 4)KSS$ on $[12] \setminus [5]$ (see Corollary 4.6) as follows, with the last row of the $(7, 4)KSS$ being duplicated.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 6 & 7 & 8 & 9 \\
2 & 6 & 7 & 10 & 11 \\
3 & 6 & 8 & 10 & 12 \\
4 & 6 & 8 & 10 & 12 \\
\end{array}
\]

The next method, $M_3$, is used if Methods $M_1$ and $M_2$ fail. The method uses a construction described in [16] to build a minimal $\binom{(m+1)}{2}, mCSS$ in $(m + 1)$ sets. This construction, called Construction $M$, is described before Method $M_3$ is given (see also Note 7.5). The proof that Construction $M$ has the properties mentioned below can be found in [16].

Construction $M$

Assume $n_1 - K = \binom{K}{2}$. A $K \times K - 1$ array $M$ can be constructed where the $K$ rows of $M$ form a $(n_1 - K, K - 1)CSS$ in which each element occurs exactly twice. Let $m_{ij}$ denote the element of $M$ in row $i$ column $j$. For each $m$, in lexicographic order, include $m$ in turn in the two positions of $M$ defined by:

\[
\min_j \min_i \{m_{ij} : m_{ij} = 0\},
\]

\[
\min_i \min_j \{m_{ij} : m_{ij} = 0\}.
\]

That is, $m$ is placed in the first row of $M$ containing 0, in the first 0-valued place in that row. $m$ is then also placed in the first column of $M$ containing 0, in the first 0-valued place in that column. This concludes Construction $M$.

Example 8.3. Let $n_1 - K = 10$ and $K - 1 = 5$. Then Construction $M$ gives
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the following array.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 5 & 6 & 7 \\
2 & 5 & 8 & 9 \\
3 & 6 & 8 & 10 \\
4 & 7 & 9 & 10 \\
\end{array}
\]

Method $M_3$ is now presented.

**Method $M_3$**

Let $n_1 = \frac{K^2 + K}{2}$ and $k_1 = K$. Then $n_2 = n - n_1 = n - \frac{K^2 + K}{2}$ and $k_2 = k - K$. Define $\mathcal{K}_1$ to be the augmented array $[\mathcal{K}_3, \mathcal{K}_4]$ where $\mathcal{K}_3$ is a $K \times 1$ array on $[K]$ and $\mathcal{K}_4$ is a $K \times (K - 1)$ array on $[\frac{K^2 + K}{2}] \setminus [K]$ formed using Construction $M$. The array $\mathcal{K}_2$ is then chosen to be a $K \times (k - K)$ array on $[n] \setminus [\frac{K^2 + K}{2}]$ where, and this is important, each element occurs at least 3 times.

(End of Method $M_3$)

It is clear that $\mathcal{K}_1$ is a $KSS$ as each element of $[n]$ in $\mathcal{K}_4$ occurs more often in $\mathcal{K}_1$ than any element of $\mathcal{K}_3$ and thus each element of $\mathcal{K}_4$ is separated from each element of $\mathcal{K}_3$. Then, provided that $\mathcal{K}_2$ is a $KSS$, it is clear that $\mathcal{K}$ is a $(n, k)KSS$ as each element of $[n]$ in $\mathcal{K}_2$ occurs more often in $\mathcal{K}$ than any element of $\mathcal{K}_1$ and so each element of $\mathcal{K}_2$ is separated from each element of $\mathcal{K}_1$.

**Note 8.1.** (1) For all cases considered in Chapter 8 and Appendix E, a $KSS \mathcal{K}_2$ in Method $M_3$ always exists. It is not known if one of Methods $M_1$, $M_2$ or $M_3$ always allows a minimal $(n, k)KSS$ to be built from smaller $KSS$s.

(2) Method $M_3$ is very easy to apply and in many cases it could be used instead of $M_1$ and $M_2$. However, it does not replace those methods. To see this, it is sufficient to try to construct a $(13, 7)KSS$ in 4 sets using $M_3$. It is quickly seen that this is impossible.
(3) To construct the array $K_2$, one has to find an appropriate placement of the elements to ensure that $K_2$ is a $KSS$. This is normally a very easy process and this is illustrated in the following example. Given the self-defining nature of $M_3$ and the ease of constructing $K_2$ in each case considered here, only the number of $p$-elements in $K_2$, $p \geq 3$, is specified when Method $M_3$ is applied in Sections 8.3 and 8.4 and Appendix E.

**Example 8.4.** A minimal $(23, 7)KSS$ in 6 sets can be constructed using Method $M_3$ as shown by

$$
\begin{align*}
1 &\ 7 &\ 8 &\ 9 &\ 10 &\ 11 &\ 22 \\
2 &\ 7 &\ 12 &\ 13 &\ 14 &\ 15 &\ 22 \\
3 &\ 8 &\ 12 &\ 16 &\ 17 &\ 18 &\ 22 \\
4 &\ 9 &\ 13 &\ 16 &\ 19 &\ 20 &\ 23 \\
5 &\ 10 &\ 14 &\ 17 &\ 19 &\ 21 &\ 23 \\
6 &\ 11 &\ 15 &\ 18 &\ 20 &\ 21 &\ 23 \\
\end{align*}
$$

A minimal $(15, 8)KSS$ in 4 sets can be constructed using Method $M_3$ with 4 3-elements and 1 4-element as shown by

$$
\begin{align*}
1 &\ 5 &\ 6 &\ 7 &\ 11 &\ 12 &\ 13 &\ 14 \\
2 &\ 5 &\ 8 &\ 9 &\ 11 &\ 12 &\ 13 &\ 15 \\
3 &\ 6 &\ 8 &\ 10 &\ 11 &\ 12 &\ 14 &\ 15 \\
4 &\ 7 &\ 9 &\ 10 &\ 11 &\ 13 &\ 14 &\ 15 \\
\end{align*}
$$

**Note 8.2.** The minimal $(15, 8)KSS$ constructed in Example 8.4 contains the element 11 in each set. It can be proved that in any minimal $(15, 8)KSS$ an element must occur in every set. This situation will be considered again in Chapters 9 and 10.

**Note 8.3.** All of the $(n, k)KSSs$ which are shown in their array representation in this chapter and Appendices F and G have been checked with a program written by C. Ramsay and with the help of P. Lieby.
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8.3 $(n, k)KSS$s which Achieve the Lower Bound in Theorem 7.3

In this section the values of $K(n, k)$, $6 \leq k \leq 12$ and $2k \leq n \leq 40$, for which $K(n, k) = \left\lceil \frac{2n}{k+1} \right\rceil$ are given. In each case a $(n, k)KSS$ is constructed using one of the methods in Section 8.2. In each case the method used when constructing the $(n, k)KSS$ is stated, and, if applicable, the minimal $(n_1, k_1)KSS$ and the minimal $(n_2, k_2)KSS$ are given. Some of the minimal $(n, k)KSS$s constructed in this way appear in Appendix F.

Note that whenever Theorem 7.9 is referred to for providing a $\binom{m+1}{2}KSS$ in $m + 1$ sets, then Construction $M$ could be used for the same purpose.

Lemma 8.1.

$$K(11, 6) = K(12, 6) = 4.$$

*Proof.* To form a $(11, 6)KSS$ in 4 sets use a minimal $(6, 3)KSS$ on $[6]$ (see Theorem 7.9) and a minimal $(5, 3)KSS$ on $[11] \setminus [6]$ (see Lemma 7.22) with $M_1$.

To form a $(12, 6)KSS$ in 4 sets use a minimal $(6, 3)KSS$ on $[6]$ and a minimal $(6, 3)KSS$ on $[12] \setminus [6]$ (see Theorem 7.9) with $M_1$.

Hence, by Theorem 7.3, $K(11, 6) = K(12, 6) = 4$. \hfill \Box

Lemma 8.2.

1. $K(13, 7) = K(14, 7) = 4.$
2. $K(17, 7) = K(18, 7) = 5.$

*Proof.* (1) To form a $(13, 7)KSS$ in 4 sets use a minimal $(7, 4)KSS$ on $[7]$ (see Corollary 4.6) and a minimal $(6, 3)KSS$ on $[13] \setminus [7]$ (see Theorem 7.9) with $M_1$. 
To form a \((14, 7)KSS\) in 4 sets use a minimal \((6, 3)KSS\) on \([6]\) (see Theorem 7.9) and a minimal \((7, 4)KSS\) on \([14] \setminus [7]\) (see Corollary 4.6) with \(M_2\).

Hence, by Theorem 7.3, \(K(13, 7) = K(14, 7) = 4\).

(2) To form a \((17, 7)KSS\) in 5 sets use a minimal \((7, 3)KSS\) on \([7]\) (see Theorem 7.10) and a minimal \((10, 4)KSS\) on \([17] \setminus [7]\) (see Theorem 7.9) with \(M_1\).

To form a \((18, 7)KSS\) in 5 sets use a minimal \((6, 2)KSS\) on \([6]\) (see Theorem 7.8) and a minimal \((11, 5)KSS\) on \([18] \setminus [7]\) (see Theorem 7.7) with \(M_2\).

Hence, by Theorem 7.3, \(K(17, 7) = K(18, 7) = 5\).

(3) To form a \((23, 7)KSS\) in 6 sets use \(M_3\) with 2 3-elements.

Hence, by Theorem 7.3, \(K(23, 7) = 6\).

\[\square\]

**Lemma 8.3.**

\(1\) \(K(15, 8) = 4\).

\(2\) \(K(19, 8) = K(20, 8) = 5\).

\(3\) \(K(23, 8) = K(24, 8) = K(25, 8) = 6\).

\(4\) \(K(30, 8) = 7\).

**Proof.** (1) To form a \((15, 8)KSS\) in 4 sets use \(M_3\) with 4 3-elements and 1 4-element as in Example 8.4.

Hence, by Theorem 7.3, \(K(15, 8) = 4\).

(2) To form a \((19, 8)KSS\) in 5 sets use a minimal \((8, 3)KSS\) on \([8]\) (see Theorem 7.10) and a minimal \((11, 5)KSS\) on \([19] \setminus [8]\) (see Theorem 7.7) with \(M_1\).

To form a \((20, 8)KSS\) in 5 sets use \(M_3\) with 5 3-elements.

Hence, by Theorem 7.3, \(K(19, 8) = K(20, 8) = 5\).

(3) To form a \((23, 8)KSS\) in 6 sets use a minimal \((7, 2)KSS\) on \([7]\) (see Theorem
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7.8) and a minimal $(15, 6)KSS$ on $[23] \setminus [8]$ (see Theorem 7.7) with $M_2$.

To form a $(24, 8)KSS$ in 6 sets use a minimal $K(7, 2)KSS$ on $[7]$ (see Theorem 7.8) and a minimal $(16, 6)KSS$ on $[24] \setminus [8]$ (see Theorem 7.7) with $M_2$.

To form a $(25, 8)KSS$ in 6 sets use $M_3$ with 4 3-elements.

Hence, by Theorem 7.3, $K(23, 8) = K(24, 8) = K(25, 8) = 6$.

(4) To form a $(30, 8)KSS$ in 7 sets use $M_3$ with 1 3-element and 1 4-element.

Hence, by Theorem 7.3, $K(30, 8) = 7$.  

Lemma 8.4.

(1) $K(21, 9) = 5$.

(2) $K(26, 9) = K(27, 9) = 6$.

(3) $K(31, 9) = K(32, 9) = 7$.

(4) $K(38, 9) = 8$.

Proof. (1) To form a $(21, 9)KSS$ in 5 sets use a minimal $(8, 3)KSS$ on $[8]$ (see Theorem 7.10) and a minimal $(12, 6)KSS$ on $[21] \setminus [9]$ (see Lemma 8.1) with $M_2$.

Hence, by Theorem 7.3, $K(21, 9) = 5$.

(2) To form a $(26, 9)KSS$ in 6 sets use $M_3$ with 4 3-elements.

To form a $(27, 9)KSS$ in 6 sets use $M_3$ with 6 3-elements.

Hence, by Theorem 7.3, $K(26, 9) = K(27, 9) = 6$.

(3) To form a $(31, 9)KSS$ in 7 sets use a minimal $(9, 2)KSS$ on $[9]$ (see Theorem 7.8 and a minimal $(22, 7)KSS$ on $[31] \setminus [9]$ (see Theorem 7.7) with $M_1$.

To form a $(32, 9)KSS$ in 7 sets use a minimal $(9, 2)KSS$ on $[9]$ (see Theorem 7.8) and a minimal $(23, 7)KSS$ on $[32] \setminus [9]$ (see Lemma 8.2) with $M_1$.


\qed
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(4) To form a $(38, 9)\text{KSS}$ in 8 sets use $M_3$ with 2 4-elements.

Hence, by Theorem 7.3, $K(38, 9) = 8$. $\blacksquare$

Lemma 8.5.

(1) $K(23, 10) = 5$.

(2) $K(28, 10) = K(29, 10) = 6$.

(3) $K(34, 10) = K(35, 10) = 7$.

(4) $K(39, 10) = K(40, 10) = 8$.

Proof. (1) To form a $(23, 10)\text{KSS}$ in 5 sets use $M_3$ with 7 3-elements and 1 4-element.

Hence, by Theorem 7.3, $K(23, 10) = 5$.

(2) To form a $(28, 10)\text{KSS}$ in 6 sets use a minimal $(10, 3)\text{KSS}$ on $[10]$ (see Theorem 7.9) and a minimal $(18, 7)\text{KSS}$ on $[28] \setminus [10]$ (see Lemma 8.2) with $M_1$.

To form a $(29, 10)\text{KSS}$ in 6 sets use $M_3$ with 8 3-elements.

Hence, by Theorem 7.3, $K(28, 10) = K(29, 10) = 6$.

(3) To form a $(34, 10)\text{KSS}$ in 7 sets use a minimal $(9, 2)\text{KSS}$ on $[9]$ (see Theorem 7.8) and a minimal $(24, 8)\text{KSS}$ on $[34] \setminus [10]$ (see Lemma 8.3) with $M_2$.

To form a $(35, 10)\text{KSS}$ in 7 sets use a minimal $(9, 2)\text{KSS}$ on $[9]$ (see Theorem 7.8) and a minimal $(25, 8)\text{KSS}$ on $[35] \setminus [10]$ (see Lemma 8.3) with $M_2$.

Hence, by Theorem 7.3, $K(34, 10) = K(35, 10) = 7$.

(4) To form a $(39, 10)\text{KSS}$ in 8 sets use a minimal $(10, 2)\text{KSS}$ on $[10]$ (see Theorem 7.8) and a minimal $(29, 8)\text{KSS}$ on $[39] \setminus [10]$ (see Theorem 7.7) with $M_1$.

To form a $(40, 10)\text{KSS}$ in 8 sets use a minimal $(10, 2)\text{KSS}$ on $[10]$ (see Theorem 7.8) and a minimal $(30, 8)\text{KSS}$ on $[40] \setminus [10]$ (see Lemma 8.3) with $M_1$. 
Hence, by Theorem 7.3, \( K(39, 10) = K(40, 10) = 8 \).

Lemma 8.6.

(1) \( K(25, 11) = 5 \).

(2) \( K(31, 11) = 6 \).

(3) \( K(37, 11) = 7 \).

Proof. (1) To form a \((25, 11)\)KSS in 5 sets use a minimal \((10, 4)\)KSS on \([11]\) (see Theorem 7.9) and a minimal \((14, 7)\)KSS on \([25] \setminus [11]\) (see Lemma 8.2) with \( M_2 \). Hence, by Theorem 7.3, \( K(25, 11) = 5 \).

(2) To form a \((31, 11)\)KSS in 6 sets use a minimal \((10, 3)\)KSS on \([10]\) (see Theorem 7.10) and a minimal \((20, 8)\)KSS on \([31] \setminus [11]\) (see Lemma 8.3) with \( M_2 \). Hence, by Theorem 7.3, \( K(31, 11) = 6 \).

(3) To form a \((37, 11)\)KSS in 7 sets use \( M_3 \) with 8 3-elements and 1 4-element. Hence, by Theorem 7.3, \( K(31, 11) = 6 \).

Lemma 8.7. \( K(33, 12) = 6 \).

Proof. To form a \((33, 12)\)KSS in 6 sets use \( M_3 \) with 12 3-elements.

8.4 \((n, k)\)KSSs which Do Not Achieve the Lower Bound in Theorem 7.3

Each \((n, k)\)KSS \( \mathcal{K} \) constructed in this section does not achieve the lower bound \( \left\lceil \frac{2n}{k+1} \right\rceil \) in Theorem 7.3. Hence the proofs first show that \( |\mathcal{K}| > \left\lceil \frac{2n}{k+1} \right\rceil \) in each case. This is most often done by using Corollary 7.3. However, some cases require other reasoning and these are considered in detail. Arguments concerning volumes are generally used in those cases (see Lemma 8.13(1) for example). In each case
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the minimal $(n_1, k_1)KSS$s and $(n_2, k_2)KSS$s are provided using the methods described in Section 8.2. This section contains only the cases for $k \leq 9$ and $n \geq 2k$ and a few special cases when Corollary 7.3 cannot be used. The rest of these cases, when $10 \leq k \leq 15$, $n \geq 2k$ and $n \leq 40$, are given in Appendix E using the same methods as described in this chapter. Some of the minimal $(n, k)KSS$s constructed in this section appear in their array representation in Appendix G.

Lemma 8.8.

(1) $K(9, 5) = 4$.

(2) $K(12, 5) = 5$.

Proof. (1) By Theorem 7.3, $K(9, 5) \geq 3$. Assume $K$ is a $(9, 5)KSS$ in 3 sets. As $|\mathcal{K}|k = 15 < 18 \leq V(\mathcal{K})$ by Theorem 7.4, $K(9, 5) > 3$ by Corollary 7.3. Example 8.1 shows a $(9, 5)KSS$ in 4 sets so $K(9, 5) = 4$.

(2) By Example 7.1, $K(12, 5) \geq 5$. Example 8.2 shows a $(12, 5)KSS$ in 5 sets so $K(12, 5) = 5$. \hfill $\square$

Note 8.4. By Lemma 7.13, Theorems 7.5 to 7.7 and Lemma 8.8, the value of $K(n, 5)$ is known for all $n$ except for $7 \leq n \leq 8$. These values will be determined in Chapter 9.

Lemma 8.9.

(1) $K(13, 6) = K(14, 6) = 5$.

(2) $K(17, 6) = 6$.

Proof. (1) By Theorem 7.3, $K(13, 6) \geq 4$ and $K(14, 6) \geq 4$. Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are a $(13, 6)KSS$ and a $(14, 6)KSS$ in 4 sets respectively. As $|\mathcal{K}_1|k = 24 < 25 \leq V(\mathcal{K}_1)$ and $|\mathcal{K}_2|k = 24 < 28 \leq V(\mathcal{K}_2)$ by Theorem 7.4, $K(13, 6) > 4$ and $K(14, 6) > 4$ by Corollary 7.3.
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To form a $(13, 6)KSS$ in $5$ sets use a minimal $(6, 3)KSS$ on $[6]$ (see Theorem 7.9) and a minimal $(7, 3)KSS$ on $[13] \setminus [6]$ (see Theorem 7.10) with $M_1$.

To form a $(14, 6)KSS$ in $5$ sets use a minimal $(6, 3)KSS$ on $[6]$ (see Theorem 7.9) and a minimal $(8, 3)KSS$ on $[14] \setminus [6]$ (see Theorem 7.10) with $M_1$.

Hence $K(13, 6) = K(14, 6) = 5$.

(2) Let $\mathcal{K}$ be a $(17, 6)KSS$ on $[17]$. By Theorem 7.3, $K(17, 6) \geq 5$. Assume $\mathcal{K}$ is a $(17, 6)KSS$ in $5$ sets. As $|\mathcal{K}|k = 30 < 36 \leq V(\mathcal{K})$ by Theorem 7.4, $K(17, 6) > 5$ by Corollary 7.3.

To form a $(17, 6)KSS$ in $6$ sets use a minimal $(5, 1)KSS$ on $[5]$ (see Lemma 7.10) and a minimal $(11, 5)KSS$ on $[17] \setminus [6]$ (see Theorem 7.7) with $M_2$. Hence $K(17, 6) = 6$. \hfill $\square$

Note 8.5. By Lemma 7.13, Theorems 7.5 to 7.7 and Lemmas 8.1 and 8.9, the value of $K(n, 6)$ is known for all $n$ except for $8 \leq n \leq 10$. These values will be determined in Chapter 9.

Lemma 8.10.

(1) $K(15, 7) = K(16, 7) = 5$.

(2) $K(19, 7) = K(20, 7) = 6$.

(3) $K(24, 7) = 7$.

Proof. (1) By Theorem 7.3, $K(15, 7) \geq 4$ and $K(16, 7) \geq 4$. Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are a $(15, 7)KSS$ and a $(16, 7)$ in $4$ sets respectively. As $|\mathcal{K}_1|k = 28 < 31 \leq V(\mathcal{K}_1)$ and $|\mathcal{K}_2|k = 28 < 34 \leq V(\mathcal{K}_2)$ by Theorem 7.4, $K(15, 7) > 4$ and $K(16, 7) > 4$ by Corollary 7.3.

To form a $(15, 7)KSS$ in $5$ sets use $(6, 3)KSS$ on $[6]$ (see Theorem 7.9) and a minimal $(8, 4)KSS$ on $[15] \setminus [7]$ (see Theorem 7.6) with $M_2$. 

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To form a $(16, 7)KSS$ in 5 sets use a minimal $(6, 3)KSS$ on $[6]$ (see Theorem 7.9) and a minimal $(9, 4)KSS$ on $[16] \setminus [7]$ (see Theorem 7.6) with $M_2$.

Hence $K(15, 7) = K(16, 7) = 5$.

(2) By Theorem 7.3, $K(19, 7) \geq 5$ and $K(20, 7) \geq 5$. Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are a $(19, 7)KSS$ and a $(20, 7)$ in 5 sets respectively. As $|\mathcal{K}_1|k = 35 < 37 \leq V(\mathcal{K}_1)$ and $|\mathcal{K}_2|k = 35 < 40 \leq V(\mathcal{K}_2)$ by Theorem 7.4, $K(19, 7) > 5$ and $K(20, 7) > 5$ by Corollary 7.3.

To form a $(19, 7)KSS$ in 6 sets use a minimal $(7, 2)KSS$ on $[7]$ (see Theorem 7.8) and a minimal $(12, 5)KSS$ on $[19] \setminus [7]$ (see Lemma 8.8) with $M_1$.

To form a $(20, 7)KSS$ in 6 sets use a minimal $(7, 2)KSS$ on $[7]$ (see Theorem 7.8) and a minimal $(13, 5)KSS$ on $[20] \setminus [7]$ (see Theorem 7.6) with $M_1$.

Hence $K(19, 7) = K(20, 7) = 6$.

(3) By Theorem 4.2, $K(24, 7) \geq 6$. Assume $\mathcal{K}$ is a $(24, 7)KSS$ in 6 sets. As $|\mathcal{K}|k = 42 < 45 \leq V(\mathcal{K})$ by Theorem 7.4, $K(24, 7) > 6$ by Corollary 7.3.

To form a $(24, 7)KSS$ in 7 sets use a minimal $(6, 1)KSS$ on $[6]$ (see Lemma 7.10) and a minimal $(17, 6)KSS$ on $[24] \setminus [7]$ (see Lemma 8.9) with $M_2$. Hence $K(24, 7) = 7$. \qed

Note 8.6. By Lemma 7.13, Theorems 7.5 to 7.7 and Lemmas 8.2 and 8.10, the value $K(n, 7)$ is known for all $n$ except for $9 \leq n \leq 12$. These values will be determined in Chapter 9.

Lemma 8.11.

1. $K(16, 8) = K(17, 8) = K(18, 8) = 5.$
2. $K(21, 8) = K(22, 8) = 6.$
Chapter 8. Particular Values of $K(n, k)$

Proof. (1) By Theorem 7.3, $K(16, 8) \geq 4$, $K(17, 8) \geq 4$ and $K(18, 8) \geq 4$.

Assume $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ are a $(16, 8)KSS$, a $(17, 8)$ and a $(18, 8)KSS$ in 4 sets respectively. As $|\mathcal{K}_1|k = 32 < 34 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 32 < 37 \leq V(\mathcal{K}_2)$ and $|\mathcal{K}_3|k = 32 < 40 \leq V(\mathcal{K}_3)$ by Theorem 7.4, $K(16, 8) > 4$, $K(17, 8) > 4$ and $K(18, 8) > 4$ by Corollary 7.3.

To form a $(16, 8)KSS$ in 5 sets use a minimal $(8, 4)KSS$ on $[8]$ and a minimal $(8, 4)KSS$ on $[16] \setminus [8]$ (see Theorem 7.6) with $M_1$ as shown in the following design.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 6 & 9 & 10 & 11 & 14 \\
1 & 2 & 4 & 7 & 9 & 10 & 12 & 15 \\
1 & 3 & 5 & 8 & 9 & 11 & 13 & 16 \\
2 & 6 & 7 & 8 & 9 & 14 & 15 & 16
\end{array}
\]

Note that $\mathcal{K}_1$ is constructed here for convenience in Lemma 9.1. The $(8, 4)KSS$ on $[8]$ used here differs from the one in Theorem 7.6 in that the element 2 is used instead of the element 1 in the last row of the $(8, 4)KSS$ on $[8]$. So that no element occurs in every set in $\mathcal{K}_1$.

To form a $(17, 8)KSS$ in 5 sets use a minimal $(8, 4)KSS$ on $[8]$ (see Theorem 7.6) and a minimal $(9, 4)KSS$ on $[17] \setminus [8]$ (see Theorem 7.6) with $M_1$.

To form a $(18, 8)KSS$ in 5 sets use a minimal $(8, 4)KSS$ on $[8]$ (see Theorem 7.6) and a minimal $(10, 4)KSS$ on $[18] \setminus [8]$ (see Theorem 7.9) with $M_1$.

Hence $K(16, 8) = K(17, 8) = K(18, 8) = 5$.

(2) By Theorem 7.3, $K(21, 8) \geq 5$ and $K(22, 8) \geq 5$. Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are a $(21, 8)KSS$ and a $(22, 8)$ in 5 sets respectively. As $|\mathcal{K}_1|k = 40 < 43 \leq V(\mathcal{K}_1)$ and $|\mathcal{K}_2|k = 40 < 46 \leq V(\mathcal{K}_2)$ by Theorem 7.4, $K(21, 8) > 5$ and $K(22, 8) > 5$ by Corollary 7.3.
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To form a $(21, 8)KSS$ in 6 sets use a minimal $(8, 3)KSS$ on $[8]$ (see Theorem 7.10) and a minimal $(13, 5)KSS$ on $[21] \setminus [8]$ (see Theorem 7.6) with $M_1$.

To form a $(22, 8)KSS$ in 6 sets use a minimal $(8, 3)KSS$ on $[8]$ (see Theorem 7.10) and a minimal $(14, 5)KSS$ on $[22] \setminus [8]$ (see Theorem 7.6) with $M_1$.

Hence $K(21, 8) = K(22, 8) = 6$.

(3) By Theorem 7.3, $K(26, 8) \geq 6$ and $K(27, 8) \geq 6$. Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are a $(26, 8)KSS$ and a $(27, 8)$ in 6 sets respectively. As $|\mathcal{K}_1|k = 48 < 51 \leq V(\mathcal{K}_1)$ and $|\mathcal{K}_2|k = 48 < 54 \leq V(\mathcal{K}_2)$ by Theorem 7.4, $K(26, 8) > 6$ and $K(27, 8) > 6$ by Corollary 7.3.

To form a $(26, 8)KSS$ in 7 sets use a minimal $(8, 2)KSS$ on $[8]$ (see Theorem 7.8) and a minimal $(18, 6)KSS$ on $[26] \setminus [8]$ (see Theorem 7.6) with $M_1$.

To form a $(27, 8)KSS$ in 7 sets use a minimal $(8, 2)KSS$ on $[8]$ (see Theorem 7.8) and a minimal $(19, 6)KSS$ on $[27] \setminus [8]$ (see Theorem 7.6) with $M_1$.

Hence $K(26, 8) = K(27, 8) = 7$.

(4) By Theorem 7.3 $K(31, 8) \geq 7$. Assume $\mathcal{K}$ is a $(31, 8)KSS$ in 7 sets. As $|\mathcal{K}|k = 56 < 58 \leq V(\mathcal{K})$ by Theorem 7.4, $K(31, 8) > 7$ by Corollary 7.3.

To form a $(31, 8)KSS$ in 8 sets use a minimal $(8, 2)KSS$ on $[8]$ (see Theorem 7.8) and a minimal $(23, 6)KSS$ on $[31] \setminus [8]$ (see Theorem 7.10) with $M_1$. Hence $K(31, 8) = 8$. □

Note 8.7. By Lemma 7.13, Theorems 7.5 to 7.7 and Lemmas 8.3 and 8.11, the value $K(n, 8)$ is known for all $n$ except for $10 \leq n \leq 14$. These values will be determined in Chapter 9.
Chapter 8. Particular Values of $K(n, k)$

Lemma 8.12.

(1) $K(18, 9) = K(19, 9) = K(20, 9) = 5$.

(2) $K(22, 9) = K(23, 9) = K(24, 9) = K(25, 9) = 6$.

(3) $K(28, 9) = K(29, 9) = K(30, 9) = 7$.

(4) $K(33, 9) = K(34, 9) = K(35, 9) = 8$.

(5) $K(39, 9) = K(40, 9) = 9$.

Proof. (1) By Theorem 7.3, $K(18, 9) \geq 4$, $K(19, 9) \geq 4$ and $K(20, 9) \geq 4$. Assume $K_1$, $K_2$ and $K_3$ are a $(18, 9)KSS$, a $(19, 9)KSS$ and a $(20, 9)KSS$ in 4 sets respectively. As $|K_1|k = 36 < 40 \leq V(K_1)$, $|K_2|k = 36 < 43 \leq V(K_2)$ and $|K_3|k = 36 < 46 \leq V(K_3)$ by Theorem 7.4, $K(18, 9) > 4$, $K(19, 9) > 4$ and $K(20, 9) > 4$ by Corollary 7.3.

To form a $(18, 9)KSS$ in 5 sets use a minimal $(9, 4)KSS$ on $[9]$ (see Theorem 7.6) and a minimal $(9, 5)KSS$ on $[18] \setminus [9]$ (see Lemma 8.8) with $M_1$.

To form a $(19, 9)KSS$ in 5 sets use a minimal $(9, 4)KSS$ on $[9]$ (see Theorem 7.6) and a minimal $(10, 5)KSS$ on $[19] \setminus [9]$ (see Theorem 7.7) with $M_1$.

To form a $(20, 9)KSS$ in 5 sets use a minimal $(9, 4)KSS$ on $[9]$ (see Theorem 7.6) and a minimal $(11, 5)KSS$ on $[20] \setminus [9]$ (see Theorem 7.7) with $M_1$.

Hence $K(18, 9) = K(19, 9) = K(20, 9) = 5$.

(2) By Theorem 7.3, $K(22, 9) \geq 5$, $K(23, 9) \geq 5$, $K(24, 9) \geq 5$ and $K(25, 9) \geq 5$. Assume $K_1$, $K_2$, $K_3$ and $K_4$ are a $(22, 9)KSS$, a $(23, 9)KSS$, a $(24, 9)KSS$ and a $(25, 9)KSS$ in 5 sets respectively. As $|K_1|k = 45 < 46 \leq V(K_1)$, $|K_2|k = 45 < 49 \leq V(K_2)$, $|K_3|k = 45 < 52 \leq V(K_3)$ and $|K_4|k = 45 < 55 \leq V(K_4)$ by Theorem 7.4, $K(22, 9) > 5$, $K(23, 9) > 5$, $K(24, 9) > 5$ and $K(25, 9) > 5$ by Corollary 7.3.
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To form a $(22, 9)KSS$ in 6 sets use a minimal $(9, 3)KSS$ on $[9]$ (see Theorem 7.10) and a minimal $(13, 6)KSS$ on $[22] \setminus [9]$ (see Lemma 8.9) with $M_1$.

To form a $(23, 9)KSS$ in 6 sets use a minimal $(9, 3)KSS$ on $[9]$ (see Theorem 7.10) and a minimal $(14, 6)KSS$ on $[23] \setminus [9]$ (see Lemma 8.9) with $M_1$.

To form a $(24, 9)KSS$ in 6 sets use a minimal $(9, 3)KSS$ on $[9]$ (see Theorem 7.10) and a minimal $(15, 6)KSS$ on $[24] \setminus [9]$ (see Theorem 7.7) with $M_1$.

To form a $(25, 9)KSS$ in 6 sets use a minimal $(9, 3)KSS$ on $[9]$ (see Theorem 7.10) and a minimal $(16, 6)KSS$ on $[25] \setminus [9]$ (see Theorem 7.7) with $M_1$.

Hence $K(22, 9) = K(23, 9) = K(24, 9) = K(25, 9) = 6$.

(3) By Theorem 7.3, $K(28, 9) \geq 6$, $K(29, 9) \geq 6$ and $K(30, 9) \geq 6$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ are a $(28, 9)KSS$, a $(29, 9)$ and $(30, 9)KSS$ in 6 sets respectively. As $|\mathcal{K}_1|k = 54 < 57 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 54 < 60 \leq V(\mathcal{K}_2)$ and $|\mathcal{K}_3|k = 54 < 63 \leq V(\mathcal{K}_3)$ by Theorem 7.4, $K(28, 9) > 6$, $K(29, 9) > 6$ and $K(30, 9) > 6$ by Corollary 7.3.

To form $(28, 9)KSS$ in 7 sets use a minimal $(9, 2)KSS$ on $[9]$ (see Theorem 7.8) and a minimal $(19, 7)KSS$ on $[28] \setminus [9]$ (see Lemma 8.10) with $M_1$.

To form a $(29, 9)KSS$ in 7 sets use a minimal $(9, 2)KSS$ on $[9]$ (see Theorem 7.8) and a minimal $(20, 7)KSS$ on $[29] \setminus [9]$ (see Lemma 8.10) with $M_1$.

To form a $(30, 9)KSS$ in 7 sets use a minimal $(9, 2)KSS$ on $[9]$ (see Theorem 7.8) $(21, 7)KSS$ on $[30] \setminus [9]$ (see Theorem 7.7) with $M_1$.

Hence $K(28, 9) = K(29, 9) = K(30, 9) = 7$.

(4) By Theorem 7.3, $K(33, 9) \geq 7$, $K(34, 9) \geq 7$ and $K(35, 9) \geq 7$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ are a $(33, 9)KSS$, $(34, 9)KSS$ and a $(35, 9)$ in 7 sets respectively. As $|\mathcal{K}_1|k = 63 < 64 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 63 < 67 \leq V(\mathcal{K}_2)$ and $|\mathcal{K}_3|k = 63 < 70 \leq V(\mathcal{K}_3)$ by Theorem 7.4, $K(33, 9) > 7$, $K(34, 9) > 7$ and $K(35, 9) > 7$ by
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Corollary 7.3.

To form a $(33,9)KSS$ in 8 sets use a minimal $(9,2)KSS$ on $[9]$ (see Theorem 7.8) and a minimal $(24,7)KSS$ on $[33] \setminus [9]$ (Lemma 8.10) with $M_1$.

To form a $(34,9)KSS$ in 8 sets use a minimal $(9,2)KSS$ on $[9]$ (see Theorem 7.8) and a minimal $(25,7)KSS$ on $[34] \setminus [9]$ (see Theorem 7.7) with $M_1$.

To form a $(35,9)KSS$ in 8 sets use a minimal $(9,2)KSS$ on $[9]$ (see Theorem 7.8) and a minimal $(26,7)KSS$ on $[35] \setminus [9]$ (see Theorem 7.7) with $M_1$.

Hence $K(33,9) = K(34,9) = K(35,9) = 8$.

(5) By Theorem 7.3, $K(39,9) \geq 8$ and $K(40,9) \geq 8$. Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are a $(39,9)KSS$ and a $(40,9)$ in 8 sets respectively. As $|\mathcal{K}_1|k = 72 < 73 \leq V(\mathcal{K}_1)$ and $|\mathcal{K}_2|k = 72 < 76 \leq V(\mathcal{K}_2)$ by Theorem 7.4, $K(39,9) > 8$ and $K(40,9) > 8$ by Corollary 7.3.

To form $(39,9)KSS$ in 9 sets use a minimal $(8,1)KSS$ on $[8]$ (see Lemma 7.10) and a minimal $(30,8)KSS$ on $[39] \setminus [9]$ (see Lemma 8.3) with $M_2$.

To form a $(40,9)KSS$ in 9 sets use a minimal $(8,1)KSS$ on $[8]$ (see Lemma 7.10) and a minimal $(31,8)KSS$ on $[40] \setminus [9]$ (see Lemma 8.11) with $M_2$.

Hence $K(39,9) = K(40,9) = 9$.  

\begin{flushright} $\Box$ \end{flushright} 

Note 8.8. By Lemma 7.13, Theorems 7.5 to 7.7 and Lemmas 8.4 and 8.12, the value $K(n,9)$ is known for all $n$ except for $11 \leq n \leq 17$. These values will be determined in Chapter 9.

A few special cases when Corollary 7.3 can not be used are now considered.
Chapter 8. Particular Values of $K(n, k)$

Lemma 8.13.

(1) $K(28, 13) = 6$.

(2) $K(29, 14) = K(30, 14) = 6$.

(3) $K(31, 15) = 6$.

Proof. (1) By Theorem 7.3, $K(28, 13) \geq 4$. Assume $\mathcal{K}$ is a $(28, 13)KSS$ in 4 sets. As $|\mathcal{K}|k = 52 < 70 \leq V(\mathcal{K})$ by Theorem 7.4, $K(28, 13) > 4$ by Corollary 7.3.

Assume $\mathcal{K}$ is a $(28, 13)KSS$ in 5 sets. As $|\mathcal{K}|k = 65$ and $V(\mathcal{K}) \geq 64$ by Theorem 7.4, Corollary 7.3 cannot apply. It can be assumed that $[13]$ is the first set in $\mathcal{K}$. At least 12 elements of the elements of $[13]$ must be covering separated in the last 4 sets of $\mathcal{K}$. By Theorem 4.2, a minimum volume minimal $(12)KSS$ has volume 22. Hence $52 - 22 = 30$ places are left to covering separate 15 elements. By Theorem 4.2, $V_{\min}(\mathcal{K}_{15}) = 32$. Therefore the 15 elements cannot be covering separated in the last 4 sets of $\mathcal{K}$. Hence $K(28, 13) > 5$.

To form a $(28, 13)KSS$ in 6 sets use a minimal $(13, 6)KSS$ on $[13]$ (see Lemma 8.9) and a minimal $(15, 7)KSS$ on $[28] \setminus [13]$ (see Lemma 8.10) with $M_1$.

Hence $K(28, 13) = 6$.

(2) By Theorem 7.3, $K(29, 14) \geq 4$ and $K(30, 14) \geq 4$. Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are $(29, 14)KSS$ and a $(30, 14)KSS$ in 4 sets. As $|\mathcal{K}_1|k = 56 < 73 \leq V(\mathcal{K}_1)$ and $|\mathcal{K}_2|k = 56 < 76 \leq V(\mathcal{K}_2)$ by Theorem 7.4, $K(29, 14) > 4$ and $K(30, 14) > 4$ by Corollary 7.3.

Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are $(29, 14)KSS$ and a $(30, 14)KSS$ in 5 sets. As $|\mathcal{K}_1|k = 70$ and $V(\mathcal{K}_1) \geq 67$, $V(\mathcal{K}_2) \geq 70$ by Theorem 7.4, Corollary 7.3 cannot apply. It can be assumed that $[14]$ is the first set in $\mathcal{K}_1$ and $\mathcal{K}_2$. At least 13 elements of the elements of $[14]$ must be covering separated in the last four sets of $\mathcal{K}_1$ and $\mathcal{K}_2$. By Theorem 4.2, a minimum volume minimal $(13)KSS$ has volume 25. Hence $56 - 25 = 31$ places in 4 sets left to covering separate 15 and 16 elements. As
Chapter 8. Particular Values of $K(n, k)$

$V_{\text{min}}(\mathcal{K}_1) = 32$ and $K(16) = 5$ by Corollary 3.2, the 15 and 16 elements cannot be covering separated in the last 4 sets. Hence $K(29, 14) > 5$ and $K(30, 14) > 5$.

To form a $(29, 14)KSS$ in 6 sets use a minimal $(13, 6)KSS$ on $[13]$ (see Lemma 8.9) and a minimal $(15, 8)KSS$ on $[29] \setminus [14]$ (see Lemma 8.3) with $M_2$.

To form a $(30, 14)KSS$ in 6 sets use a minimal $(13, 6)KSS$ on $[13]$ (see Lemma 8.9) and a minimal $(16, 8)KSS$ on $[30] \setminus [14]$ (see Lemma 8.3) with $M_2$.

Hence $K(29, 14) = K(30, 14) = 6$.

(3) By Theorem 7.3, $K(31, 15) \geq 4$, $K(32, 15) \geq 4$. Assume $\mathcal{K}$ is a $(31, 15)KSS$ in 4 sets. As $|\mathcal{K}|k = 60 < 82 \leq V(\mathcal{K})$ by Theorem 7.4, $K(31, 15) > 4$ by Corollary 7.3.

Assume $\mathcal{K}$ is a $(31, 15)KSS$ in 5 sets. As $|\mathcal{K}|k = 75$ and $V(\mathcal{K}) \geq 73$ by Theorem 7.4, Corollary 7.3 cannot apply. It can be assumed that $[15]$ is the first set in $\mathcal{K}$. At least 14 of the elements of $[15]$ must be covering separated in the last 4 sets of $\mathcal{K}$. By Theorem 4.2, a minimum volume minimal $(14)KSS$ has volume 28. Hence $60 - 28 = 32$ places in 4 sets left to covering separate 16 elements. By Corollary 3.2, $K(16) = 5$. Therefore the 16 elements cannot be covering separated in the last 4 sets of $\mathcal{K}$. Hence $K(31, 15) > 5$.

To form a $(31, 15)KSS$ in 6 sets use a minimal $(14, 7)KSS$ on $[14]$ (see Lemma 8.2) and a minimal $(16, 8)KSS$ on $[31] \setminus [15]$ (see Lemma 8.11) with $M_2$.

Hence $K(31, 15) = 6$. 

\qed
Chapter 9

More on Minimal \((n, k)\) Covering Separating Systems
9.1 Introduction

Two related conjectures are stated in Section 9.2 and they are shown to be valid in many cases. It is also shown how to determine $K(n, k)$ when $k \leq 15$ and $n < 2k$. Section 9.3 contains a table which gives the value of $K(n, k)$ for each $2 \leq n \leq 40$ and $1 \leq k \leq \min\{15, n-1\}.$

9.2 Two Conjectures

The following conjectures were formulated whilst proving Theorems 7.5 to 7.7, and Lemmas 8.1 to 8.13 and Lemmas E.1 to E.6.

Conjecture 9.1. For all $k$ and $n$ with $n \geq 2k$, there exists a minimal $(n, k)KSS$ in which no element of $[n]$ occurs in every set.

The conjecture has been partly proved as stated in the following lemma.

Lemma 9.1. For all $k$ and $n$ with $1 \leq k \leq 15$ and $n \geq 2k$ with the exception of cases when $41 \leq n < \frac{k^2}{2}$, there exists a minimal $(n, k)KSS$ in which no element of $[n]$ occurs in every set.

Note 9.1. Lemma 9.1 says that Conjecture 9.1 holds for values of $n$ and $k$ such that $1 \leq k \leq 15$, $n \geq 2k$, and $n \leq 40$ or $n \geq \frac{k^2}{2}$. Conjecture 9.1 is known to be true for some values of $n$ and $k$ when $1 \leq k \leq 15$, $n \geq 2k$, and $41 \leq n < \frac{k^2}{2}$. In particular it is known that Conjecture 9.1 holds for all $n \geq 2k$ when $k = 10$ or $k = 11$. The proof of this is not included here.

Proof of Lemma 9.1. By the construction of the $(n, k)KSS$s in the proofs of Theorems 7.5 and 7.6, and Lemmas 8.1 to 8.13 and Lemmas E.1 to E.6 (see Appendix E), there exists a $(n, k)KSS$ $\mathcal{K}$ in which there is no element of $[n]$ occurring in every set of $\mathcal{K}$ except in the $(4, 2)KSS$, $(5, 2)KSS$ and $(8, 4)KSS$.
cases. However, the lemma can be seen to be true in these cases by taking 
\{12, 24, 34\}, \{12, 13, 24, 25\} and \{1236, 1247, 1358, 2678\} as a minimal \((4, 2)KSS\), 
\((5, 2)KSS\) and \((8, 4)KSS\) respectively.

The following conjecture is a variant of Conjecture 9.1 and is stated here because of its direct reference to \(K(n, k)\).

**Conjecture 9.2 (Near Symmetry Conjecture).** For all \(k\) and \(n\) with \(n \geq 2k\), \(K(n+1, n+1-k) = K(n, k)\).

The near symmetry conjecture is valid at least for the values stated in the following theorem.

**Theorem 9.1.** For all \(k\) and \(n\) with \(1 \leq k \leq 15\) and \(n \geq 2k\), with the exception of cases when \(41 \leq n < \frac{k^2}{2}\), 
\[K(n+1, n+1-k) = K(n, k)\]

**Note 9.2.** (1) Theorem 9.1 says that Conjecture 9.2 holds for values of \(n\) and \(k\) such that \(1 \leq k \leq 15\), \(n \geq 2k\), and \(n \leq 40\) or \(n \geq \frac{k^2}{2}\). As for Conjecture 9.1, Conjecture 9.2 is also known to be true for some values of \(n\) and \(k\) when \(1 \leq k \leq 15\), \(n \geq 2k\), and \(41 \leq n < \frac{k^2}{2}\) and in particular, when \(k = 10\) or \(k = 11\). The proof of this is not included here.

(2) The truth of Conjecture 9.1 would mean that Conjecture 9.2 would also be true and that Theorem 9.1 could be extended to hold for all values of \(n \geq 2k\).

**Proof of Theorem 9.1.** By Lemma 9.1, there exists a minimal \((n, k)KSS\) in which no element of \([n]\) occurs in every set for each value of \(k\) and \(n\) allowed in the theorem. Let \(\mathcal{K}\) be such a minimal \((n, k)KSS\). By Lemma 7.9, the complement \(\mathcal{K}'\) of \(\mathcal{K}\) is a \((n, n-k)KSS\). Now construct a collection \(\mathcal{K}''\) from \(\mathcal{K}'\) by including the element \(n+1\) in each member of \(\mathcal{K}'\). It is clear that \(\mathcal{K}''\) is a \((n+1, n+1-k)KSS\) since under this process the covering separation property on \([n]\) is still preserved.
and the element \( n + 1 \) is separated from each element of \([n]\) by \( A \cup \{n + 1\} \), \( A \in \mathcal{K}' \). Hence \( K(n + 1, n + 1 - k) \leq K(n, k) \).

Conversely, taking the complement of \( \mathcal{K}'' \) gives a \((n, k)KSS\) since the element \( n + 1 \) occurs in every set of \( \mathcal{K}'' \). Hence \( K(n, k) \leq K(n + 1, n + 1 - k) \).

Therefore \( K(n + 1, n + 1 - k) = K(n, k) \). \( \square \)

**Note 9.3.** The assumption that \( n \geq 2k \) in Conjecture 9.1, Lemma 9.1, Conjecture 9.2 and Theorem 9.2 is necessary. To see this recall that Lemma 7.12 shows that a minimal \((n, n - 1)KSS\) \( \mathcal{K} \) must have an element which occurs in every set in \( \mathcal{K} \).

This is not an isolated case. For example, in Note 8.2 it is stated that a minimal \((15, 8)KSS\) \( \mathcal{K} \) must have an element which occurs in every set in \( \mathcal{K} \).

### 9.3 Table of the Values of \( K(n, k) \) for \( 2 \leq n \leq 40 \) and \( 1 \leq k \leq \min\{15, n - 1\} \)

Using the results in this chapter and those in Chapters 7 and 8 and Appendix E, the exact values of \( K(n, k) \) are shown in Table 9.1 for \( k \leq 15 \) and \( 2 \leq n \leq 40 \). Table 9.1 also provides the values of \( S(n, k) \) for \( k \leq 15 \) and \( 2 \leq n \leq 40 \) through an application of Lemma 7.7.

To read the table, note that the terms written in the form \([.\] are the diagonal elements in the table. That is, they are in the positions in table in which \( n = 2k \). The values in the positions to the right of the diagonal in each row come from Theorem 9.1.
Table 9.1: Known values of $K(n, k)$ for $2 \leq n \leq 40$ and $1 \leq k \leq \min\{15, n-1\}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
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<td>14</td>
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<td>15</td>
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</table>

Note: The table entries represent the smallest $K(n, k)$ for given $n$ and $k$. The values are known up to $n = 40$ and $k \leq \min\{15, n-1\}$.
Chapter 10

Covering Separating Systems
and Finite Topologies

This chapter introduces one area of research in which this work on covering separating systems may find an application, namely in the study of finite topological spaces. This application is speculative but felt to be worth including as a possible future project.

The chapter begins with a basic review of finite topologies. The review is sufficient to understand the connections being made, and does not present the most general view of topologies. For example, the concept of open sets and closed sets in a topology is not considered here. The chapter continues with some basic results on the enumeration of finite topologies and points out the discrepancies in certain values derived by various authors. The fact that there has been discrepancies in the enumeration problems provides some motivation to consider finite topologies from a new perpective. Covering separating systems may prove useful in doing this.

**Definition 10.1.** A collection of sets $\mathcal{T}$ on $[n]$ is said to be **union-closed** if $A \cup B \in \mathcal{T}$ whenever $A, B \in \mathcal{T}$. A collection of sets $\mathcal{T}$ on $[n]$ is said to be
intersection-closed if $A \cap B \in \mathcal{T}$ whenever $A, B \in \mathcal{T}$. The union-closure of a collection of sets $\mathcal{C}$ is denoted by $\bigcup \mathcal{C}$. The intersection-closure of a collection of sets $\mathcal{C}$ is denoted by $\bigcap \mathcal{C}$. If $\mathcal{T}$ is union-closed and intersection-closed and $[n], \emptyset \in \mathcal{T}$ then $\mathcal{T}$ is called a topology or topological space on $[n]$. A base for a topology $\mathcal{T}$ is a collection of subsets of $\mathcal{T}$ whose union-closure is $\mathcal{T}$. A subbase for a topology $\mathcal{T}$ is a collection of subsets of $\mathcal{T}$ whose intersection-closure is a base for $\mathcal{T}$. The discrete topology on $[n]$ is the topology which contains all of the subsets of $[n]$. That is, the discrete topology corresponds to the power set on $[n]$.

More details on topological spaces can be found in [13] or [15].

**Example 10.1.** Consider the collection of sets $S = \{123, 145, 235\}$. The intersection-closure of $S$ is $\bigcap S = \{\emptyset, 1, 5, 23, 123, 145, 235\}$. The union-closure of $\bigcap S$ is the collection $\mathcal{T} = \{\emptyset, 1, 5, 23, 123, 145, 235, 1235, 12345\}$. $\mathcal{T}$ is union-closed and intersection-closed and contains $\emptyset$ and $[5]$. Thus $\mathcal{T}$ is a topology on $[5]$. $S$ is a subbase for $\mathcal{T}$ and $\bigcap S$ is a basis for $\mathcal{T}$.

Finite topological spaces can be classified into various types as defined below.

**Definition 10.2.** A $T_0$-space (or $T_0$-topology) $\mathcal{T}$ is a topological space in which for each pair of elements $a, b \in [n]$ there is a set in $\mathcal{T}$ which contains $a$ but not $b$ or vice versa. A $T_1$-space is a topological space $\mathcal{T}$ in which for each pair of elements $a, b \in [n]$ there is a set in $\mathcal{T}$ which contains $a$ but not $b$ and vice versa.

Clearly a $T_0$-space on $[n]$ corresponds to a covering separating system which includes $\emptyset$ and a $T_1$ on $[n]$ space corresponds to a completely separating system. Further, a collection of sets $\mathcal{S}$ is a subbase for a $T_0$-topology on $[n]$ if and only if $\mathcal{S}$ is a covering separating system in which no element occurs in every set in $\mathcal{S}$. Similarly, a collection of sets $\mathcal{S}$ is a subbase for a $T_1$-topology on $[n]$ if and only if $\mathcal{S}$ is a completely separating system. This is formalised in the following lemmas.
Chapter 10. Covering Separating Systems and Finite Topologies

Lemma 10.1. Let \( C \subseteq \mathcal{P}([n]), n > 1 \). \( C \) is a completely separating system if and only if \( C \) is a subbsase for the discrete topology \( \mathcal{D} \) on \([n]\).

Proof. Assume \( C \) is a CSS. Then \( \{i\} = \cap \{A \in C : i \in A\} \). This is true for each \( i \in [n] \) so \( \cap C \) includes all singletons and thus \( \emptyset \). Hence \( C \) is a subbsase for the discrete topology \( \mathcal{D} \).

Conversely, assume \( C \) is a subbsase for the discrete topology \( \mathcal{D} \). Suppose \( C \) is not a CSS. Then there exist two distinct elements \( i, j \) of \([n]\) such that \( i \) is not separated from \( j \). That is, if there is a member of \( C \) containing \( i \) then it also contains \( j \). Hence the set \( \cap C \) does not contain the singleton \( \{i\} \). Thus \( C \) is not a subbsase for the discrete topology \( \mathcal{D} \). This contradicts the assumption. Therefore \( C \) is a CSS. \( \square \)

Corollary 10.1. A subcollection \( C \subseteq \mathcal{P}([n]), n > 1 \), is a CSS if and only if \( \bigcap C \) contains all singletons of \([n]\).

Lemma 10.2. Let \( S \subseteq \mathcal{P}([n]) \). \( S \) is a covering separating system in which no element occurs in every set if and only if \( S \) is a subbsase for a \( T_0 \)-space.

Proof. Assume that \( S \) is \((n)KSS \) in which no elements occurs in every set. Then \( \emptyset \notin \bigcap S, S \subseteq \mathcal{T} = \bigcup(\bigcap S) \) and \([n] \in \mathcal{T} \). Thus \( \mathcal{T} \) is a \( T_0 \)-space.

Conversely, assume that \( S \) is a subbsase for a \( T_0 \)-space \( \mathcal{T} \) on \([n]\). Then \( \bigcup S = \mathcal{T} \). To show that \( S \) is a \((n)KSS \), it is necessary to show that for each \( a, b \in [n] \) there exists a set \( A \in S \) with \( a \in A, b \notin A \) or vice versa. To do this, assume that there exists \( a, b \in [n] \) for which this condition is not met. Then this implies that whenever \( a \in T \in \mathcal{T}, b \in T \in \mathcal{T} \) and vice versa. Then \( a \) and \( b \) are not separated from one another in \( \mathcal{T} \) and so \( \mathcal{T} \) is not a \( T_0 \)-space. It follows that \( S \) is a \((n)KSS \). \( \square \)

Note 10.1. (1) There is only one \( T_1 \)-topology on a finite set, namely the discrete topology. However, there are many \( T_0 \)-topologies on \([n]\) for most values of \( n \).
(2) A minimal $KSS$ is a minimal subbase for a $T_0$-space, but the minimal subbase for a $T_0$-space is not always a minimal $KSS$. For example $S = \{1, 12, 123\}$ is a minimal subbase for the $T_0$-space $\{\emptyset, 1, 12, 123\}$ on $[3]$ but $S$ is not a minimal $KSS$.

A difficult problem in the study of finite topologies is to determine the number of distinct labelled topologies for each $[n]$, or to determine the number of non-isomorphic (unlabelled topologies) for each $[n]$. Various authors have attempted this for small $n$. In Table 10.1 the number of labelled topologies and the number of labelled $T_0$-topologies for each $n \leq 14$ is given. These values are a compilation of the latest results in [23], [10], and [8]. In Table 10.2 the number of non-isomorphic topologies for each $n \leq 5$ is given. These values are taken from [6] and [24]. The difficulty of the problem is reflected in several ways.

The small number of $n$ for which the values are known provides an indication of the computational complexity of the problem. Moreover, the values in Table 10.1 are the most recently obtained values and several of them differ from previously obtained values. In particular, the number of labelled topologies for $n = 5, 6$ and 8 have been modified over time.

For example, the values 7181 and 145807 for the number of labelled topologies when $n = 5$ and 6 respectively appeared in [23]. In [10] 64655994 was given as the number of labelled topologies when $n = 8$.

A motivation for studying covering separating systems is that it may help to classify $T_0$-topologies according to characteristics of their subbases and thus aid in the study of $T_0$-topologies. This, in turn, may be useful in the determination of the number of $T_0$-topologies on $[n]$ or the number of non-isomorphic topologies on $[n]$. This application is beyond the scope of the thesis.
### Chapter 10. Covering Separating Systems and Finite Topologies

<table>
<thead>
<tr>
<th>( n )</th>
<th>Number of labelled topologies</th>
<th>Number of labelled ( T_0 )-spaces</th>
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<tbody>
<tr>
<td>1</td>
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<td>6 942</td>
<td>4 231</td>
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<tr>
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<td>6 129 859</td>
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<td>171 850 728 381 587 059 351</td>
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<td>115 617 051 977 054 267 807 460</td>
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Table 10.1: Number of labelled topologies and labelled \( T_0 \)-spaces

<table>
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<th>( n )</th>
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Table 10.2: Number of non-isomorphic topologies
Appendix A

List of Standard Notation and Terminology
Appendix A. List of Standard Notation and Terminology

\[ \mathbb{Z}^+ \] The set of positive integers

\[ [n] \] The set \{1, 2, \ldots, n\}

\[ |A| \] The cardinality (size) of the set \( A \)

\( A' \) The complement of the set \( A \)

\( A \setminus B \) The set difference \( = \{ a \in A : a \notin B \} \)

\( C' \) The complement of the collection \( C \)

\( C_1 \cong C_2 \) \( C_1 \) is isomorphic to \( C_2 \)

\( V(G) \) The vertex-set of a graph \( G \)

\( E(G) \) The edge-set of a graph \( G \)

\( \lfloor x \rfloor \) The floor function of \( x \)

\( \lceil x \rceil \) The ceiling function of \( x \)

\( \binom{n}{k} \) The binomial coefficient

\( R(n) \sim S(n) \) \( R(n) \) is asymptotic to \( S(n) \)
Appendix B

Catalogues of Non-Isomorphic Minimal \((n)KSSs\) and \((n)SSs\)
Appendix B. Catalogues of Non-Isomorphic Minimal \((n)KSSs\) and \((n)SSs\)

B.1 Catalogues of Non-Isomorphic Minimal \((n)KSSs\) for \(2 \leq n \leq 7\)

(1) \(n = 2\): \(K = 2\).

\[
\begin{array}{cccc}
1 & 2 \\
2 & 1 \\
\end{array}
\]

\(K_{2,1} \cong K_{2,2}'\).

(2) \(n = 3\): \(K = 2\).

\[
\begin{array}{cccc}
1 & 2 \\
1 & 3 \\
\end{array}
\]

(3) \(n = 4\): \(K = 3\).

\[
\begin{array}{cccc}
1 & 2 & 1 & 2 \\
1 & 3 & 4 & 1 \\
2 & 4 & 1 & 4 \\
1 & 2 & 3 & 1 \\
1 & 2 & 3 & 1 \\
1 & 2 & 4 & 1 \\
1 & 4 & 1 & 3 \\
1 & 2 & 3 & 1 \\
1 & 3 & 4 & 1 \\
\end{array}
\]

Here \(K_{4,3} \cong K_{4,3}', K_{4,4} \cong K_{4,4}', K_{4,5} \cong K_{4,5}'\) and \(K_{4,6} \cong K_{4,6}'\).
(4) $n = 5$: $K = 3$.

$1 \ 2 \ 3 \quad 1 \ 2 \ 3 \quad 1 \ 2 \ 3 \quad 1 \ 2 \ 3$

$\mathcal{K}_{5,1}: 1 \ 4 \quad , \quad \mathcal{K}_{5,2}: 1 \ 4 \ 5 \quad , \quad \mathcal{K}_{5,3}: 1 \ 2 \ 4 \quad , \quad \mathcal{K}_{5,4}: 1 \ 2 \ 4 \quad ,$

$2 \ 5 \quad 2 \ 4 \quad 1 \ 5 \quad 1 \ 3 \ 5$

$1 \ 2 \ 3 \ 4 \quad 1 \ 2 \ 3 \ 4$

$\mathcal{K}_{5,5}: 1 \ 2 \ 5 \quad , \quad \mathcal{K}_{5,6}: 1 \ 2 \ 5 \quad .$

$1 \ 3 \quad 1 \ 3 \ 5$

Here $\mathcal{K}_{5,2} \cong \mathcal{K}_{5,1}$.

(5) $n = 6$: $K = 3$.

$1 \ 2 \ 3$

$\mathcal{K}_{6,1}: 1 \ 4 \ 5 \quad ,$

$2 \ 4 \ 6$

$1 \ 2 \ 3 \ 4 \quad 1 \ 2 \ 3 \ 4$

$\mathcal{K}_{6,2}: 1 \ 2 \ 5 \quad , \quad \mathcal{K}_{6,3}: 1 \ 2 \ 5 \ 6 \ .$

$1 \ 3 \ 6 \quad 1 \ 3 \ 6$

Here $\mathcal{K}_{6,1} \cong \mathcal{K}_{6,1}$.

(6) $n = 7$: $K = 3$.

$1 \ 2 \ 3 \ 4$

$\mathcal{K}_{7,1}: 1 \ 2 \ 5 \ 6 \ .$

$1 \ 3 \ 5 \ 7$
B.2 Catalogues of Non-Isomorphic Minimal $(n)SS$s for $2 \leq n \leq 8$

(1) $n = 2$: $S = 1$.

$S_{2,1}: \begin{array}{c}
1 \\
\end{array}$

Here $S_{2,1} \cong S'_{2,1}$.

(2) $n = 3$: $S = 2$.

$S_{3,1} = K_{2,1}: \begin{array}{c}
1 & 2 \\
2 & , \\
\end{array}$ $S_{3,2} = K_{2,2}: \begin{array}{c}
1 & 2 \\
, \\
\end{array}$

$S_{3,3} = K_{3,1}: \begin{array}{c}
1 & 2 \\
, \\
1 & 3 \\
\end{array}$

Here $S_{3,3} \cong S'_{3,1}$ and $S_{3,2} \cong S'_{3,2}$.

(3) $n = 4$: $S = 2$.

$S_{4,1} = K_{3,1}: \begin{array}{c}
1 & 2 \\
, \\
1 & 3 \\
\end{array}$

Here $S_{4,1} \cong S'_{4,1}$.

(4) $n = 5$: $S = 3$.

\[
\begin{array}{ccc}
1 & 2 & 1 & 2 \\
S_{5,1} = K_{4,1}: & 3 & , & S_{5,2} = K_{4,2}: & 1 & 3 \\
& , & S_{5,3} = K_{4,3}: & 1 & 3 & , \\
& 4 & 1 & 4 & 2 & 4 \\
\end{array}
\]
Appendix B. Catalogues of Non-Isomorphic Minimal $(n)KSS$s and $(n)SS$s

| $S_{5,4} = K_{4,4} : 1 4$ | $S_{5,5} = K_{4,5} : 1 4$ | $S_{5,6} = K_{4,6} : 1 2$ |
| $2$ | $2 4$ | $1 4$ |
| $1 2 3$ | $1 2 3$ | $1 2 3$ |

$S_{5,7} = K_{4,7} : 1 2 4$, $S_{5,8} = K_{4,8} : 1 2 4$, $S_{5,9} = K_{4,9} : 1 2 4$, $S_{5,10} = K_{4,10} : 1 2$, $1 3$

| $S_{5,11} = K_{5,1} : 1 4$ | $S_{5,12} = K_{5,2} : 1 4 5$ |
| $2 5$ | $2 4$ |
| $1 2 3$ | $1 2 3$ |

$S_{5,13} = K_{5,3} : 1 2 4$, $S_{5,14} = K_{5,4} : 1 2 4$, $S_{5,15} = K_{5,5} : 1 2 5$, $S_{5,16} = K_{5,6} : 1 2 5$

| $1 2 3 4$ | $1 2 3 4$ |

Here $S_{5,1} \cong S_{5,16}^t$, $S_{5,2} \cong S_{5,9}^t$, $S_{5,3} \cong S_{5,14}^t$, $S_{5,4} \cong S_{5,15}^t$, $S_{5,5} \cong S_{5,13}^t$, $S_{5,6} \cong S_{5,8}^t$,

$S_{5,7} \cong S_{5,10}^t$, $S_{5,11} \cong S_{5,12}^t$.

(5) $n = 6$: $S = 3$.

| $S_{6,1} = K_{6,1} : 1 4$ | $S_{6,2} = K_{6,2} : 1 4 5$ |
| $2 5$ | $2 4$ |
Appendix B. Catalogues of Non-Isomorphic Minimal \((n)\mathbb{K}SSs\) and \((n)SSs\)

\[\begin{array}{c|c}
1 & 2 & 3 \\
\hline
S_{6,3} = \mathbb{K}_{5,3} : & 1 & 2 & 4 \\
& 1 & 5 \\
S_{6,4} = \mathbb{K}_{5,4} : & 1 & 2 & 4 \\
& 1 & 3 & 5 \\
S_{6,5} = \mathbb{K}_{5,5} : & 1 & 2 & 5 \\
& 1 & 3 \\
S_{6,6} = \mathbb{K}_{5,6} : & 1 & 2 & 5 \\
& 1 & 3 & 5 \\
S_{6,7} = \mathbb{K}_{6,1} : & 1 & 4 & 5 \\
& 2 & 4 & 6 \\
S_{6,8} = \mathbb{K}_{6,2} : & 1 & 2 & 5 \\
& 1 & 3 & 6 \\
S_{6,9} = \mathbb{K}_{6,3} : & 1 & 2 & 5 \\
& 1 & 3 & 6 \\
\end{array}\]

Here \(S_{6,1} \cong S'_{6,9}, S_{6,2} \cong S'_{6,8}, S_{6,3} \cong S'_{6,6}, S_{6,4} \cong S'_{6,4}, S_{6,5} \cong S'_{6,5}, S_{6,7} \cong S'_{6,7}\).

\[(6)\ n = 7: S = 3.\]

\[\begin{array}{c|c}
1 & 2 & 3 \\
\hline
S_{7,1} = \mathbb{K}_{6,1} : & 1 & 4 & 5 \\
& 2 & 4 & 6 \\
S_{7,2} = \mathbb{K}_{6,2} : & 1 & 2 & 5 \\
& 1 & 3 & 6 \\
S_{7,3} = \mathbb{K}_{6,3} : & 1 & 2 & 5 \\
& 1 & 3 & 6 \\
S_{7,4} = \mathbb{K}_{7,1} : & 1 & 2 & 5 \\
& 1 & 3 & 5 & 7 \\
\end{array}\]

Here \(S_{7,1} \cong S'_{7,4}, S_{7,2} \cong S'_{7,3}\).
(7) \( n = 8 \): \( S = 3 \).

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
S_{8,1} : & 1 & 2 & 5 & 6 \\
1 & 3 & 5 & 7 \\
\end{array}
\]

Here \( S_{8,1} \cong S_{8,1}^* \).
Appendix C

Examples for Theorem 7.6
As examples of the construction used in the proof of Theorem 7.6, consider the following arrays, the rows of which are minimal \((n, k)\)KSSs. Highlighted rows and columns are appended to the previous minimal \((n, k)\)KSSs.

### C.1 Case: \(k = 3, \; n = 5\)

\[
\begin{array}{ccc}
1 & 2 & 3 \\
K(5, 3) = 3, & 1 & 2 & 4 \\
& 1 & 3 & 5
\end{array}
\]

### C.2 Case: \(k = 4, \; 8 \leq n \leq 9\)

\[
\begin{array}{cccc}
1 & 2 & 3 & 6 \\
K(8, 4) = 4, & 1 & 2 & 4 & 7 \\
& 1 & 3 & 5 & 8 \\
& 6 & 7 & 8 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 6 \\
K(9, 4) = 4, & 1 & 2 & 4 & 7 \\
& 1 & 3 & 5 & 8 \\
& 6 & 7 & 8 & 9
\end{array}
\]
Appendix C. Examples for Theorem 7.6

C.3 Case: $k = 5$, $13 \leq n \leq 14$

\[
\begin{array}{cccc}
1 & 2 & 3 & 6 \\
1 & 2 & 4 & 7 \\
K(13,5) = 5, & 1 & 3 & 5 & 8 \\
6 & 7 & 8 & 1 \\
9 & 10 & 11 & 12 & 13
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 6 \\
1 & 2 & 4 & 7 \\
K(14,5) = 5, & 1 & 3 & 5 & 8 \\
6 & 7 & 8 & 9 \\
10 & 11 & 12 & 13 & 14
\end{array}
\]

C.4 Case: $k = 6$, $18 \leq n \leq 20$

\[
\begin{array}{cccc}
1 & 2 & 3 & 6 \\
1 & 2 & 4 & 7 \\
K(18,6) = 6, & 1 & 3 & 5 & 8 \\
6 & 7 & 8 & 1 \\
9 & 10 & 11 & 12 & 13 & 18 \\
14 & 15 & 16 & 17 & 18 & 1
\end{array}
\]
### Appendix C. Examples for Theorem 7.6

#### $K(19, 6) = 6,$

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 6 & 9 & 14 \\
1 & 2 & 4 & 7 & 10 & 15 \\
1 & 3 & 5 & 8 & 11 & 16 \\
6 & 7 & 8 & 1 & 12 & 17 \\
9 & 10 & 11 & 12 & 13 & 18 \\
14 & 15 & 16 & 17 & 18 & 19 \\
\end{array}
\]

#### $K(20, 6) = 6,$

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 6 & 10 & 15 \\
1 & 2 & 4 & 7 & 11 & 16 \\
1 & 3 & 5 & 8 & 12 & 17 \\
6 & 7 & 8 & 9 & 13 & 18 \\
10 & 11 & 12 & 13 & 14 & 19 \\
15 & 16 & 17 & 18 & 19 & 20 \\
\end{array}
\]

#### C.5 Case: $k = 7, 25 \leq n \leq 27$

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 6 & 9 & 14 & 19 \\
1 & 2 & 4 & 7 & 10 & 15 & 20 \\
1 & 3 & 5 & 8 & 11 & 16 & 21 \\
6 & 7 & 8 & 1 & 12 & 17 & 22 \\
9 & 10 & 11 & 12 & 13 & 18 & 23 \\
14 & 15 & 16 & 17 & 18 & 1 & 24 \\
19 & 20 & 21 & 22 & 23 & 24 & 25 \\
\end{array}
\]
Appendix C. Examples for Theorem 7.6

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<tbody>
<tr>
<td>1 2 4 7 10 15 21</td>
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<tr>
<td>1 3 5 8 11 16 22</td>
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</table>

\[ K(26, 7) = 7, \]

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<td>14 15 16 17 18 19 25</td>
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\[ K(27, 7) = 7, \]

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<td>21 22 23 24 25 26 27</td>
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C.6 Case: \( k = 8, 32 \leq n \leq 35 \)

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\[ K(32, 8) = 8, \]

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<td>14 15 16 17 18 1 24</td>
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<td>19 20 21 22 23 24 25</td>
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<td>26 27 28 29 30 31 32 1</td>
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</tbody>
</table>
### Appendix C. Examples for Theorem 7.6

| $K(33, 8) = 8$ | 1 2 3 6 9 14 19 26 |
|               | 1 2 4 7 10 15 20 27 |
|               | 1 3 5 8 11 16 21 28 |
|               | 6 7 8 1 12 17 22 29 |
|               | 9 10 11 12 13 18 23 30 |
|               | 14 15 16 17 18 1 24 31 |
|               | 19 20 21 22 23 24 25 32 |
|               | 26 27 28 29 30 31 32 33 |

| $K(34, 8) = 8$ | 1 2 3 6 9 14 20 27 |
|               | 1 2 4 7 10 15 21 28 |
|               | 1 3 5 8 11 16 22 29 |
|               | 6 7 8 1 12 17 23 30 |
|               | 9 10 11 12 13 18 24 31 |
|               | 14 15 16 17 18 19 25 32 |
|               | 20 21 22 23 24 25 26 33 |
|               | 27 28 29 30 31 32 33 34 |

| $K(35, 8) = 8$ | 1 2 3 6 10 15 21 28 |
|               | 1 2 4 7 11 16 22 29 |
|               | 1 3 5 8 12 17 23 30 |
|               | 6 7 8 9 13 18 24 31 |
|               | 10 11 12 13 14 19 25 32 |
|               | 15 16 17 18 19 20 26 33 |
|               | 21 22 23 24 25 26 27 34 |
|               | 28 29 30 31 32 33 34 35 |
Appendix D

Examples for Theorem 7.7
As examples of the construction used in the proof of Theorem 7.7, consider the following arrays, the rows of which are minimal \((n,k)\)KSSs. Highlighted rows and columns are appended to the previous minimal \((n,k)\)KSSs.

**D.1 Case: \(k = 5, n = 10,11\)**

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 7 \\
1 & 2 & 3 & 5 & 8 \\
2 & 4 & 5 & 6 & 9 \\
3 & 4 & 5 & 6 & 10 \\
\end{array}
\]

\(K(10,5) = 4,\)

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 8 \\
1 & 2 & 5 & 6 & 9 \\
1 & 3 & 5 & 7 & 10 \\
8 & 9 & 10 & 11 & 2 \\
\end{array}
\]

**D.2 Case: \(k = 6, n = 15,16\)**

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 7 & 11 \\
1 & 2 & 3 & 5 & 8 & 12 \\
2 & 4 & 5 & 6 & 9 & 13 \\
3 & 4 & 5 & 6 & 10 & 14 \\
11 & 12 & 13 & 14 & 15 & 1 \\
\end{array}
\]

\(K(15,6) = 5,\)
Appendix D. Examples for Theorem 7.7

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\[ K(16, 6) = 5, \]

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\[ K(16, 6) = 5, \]

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**D.3 Case: \( k = 7, n = 21, 22 \)**

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\[ K(21, 7) = 6, \]

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\[ K(21, 7) = 6, \]

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\[ K(21, 7) = 6, \]

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\[ K(22, 7) = 6, \]

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\[ K(22, 7) = 6, \]

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\[ K(22, 7) = 6, \]

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### D.4 Case: $k = 8, n = 28, 29$

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Appendix D. Examples for Theorem 7.7

D.5 Case: $k = 9, n = 36, 37$

|       | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $K(36, 9) = 8,$ |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |10  |11  |12  |13  |14  |15  |16  |17  |18  |19  |20  |21  |22  |23  |24  |25  |26  |27  |28  |29  |30  |31  |32  |33  |34  |35  |36  |37  |
|       | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |10  |11  |12  |13  |14  |15  |16  |17  |18  |19  |20  |21  |22  |23  |24  |25  |26  |27  |28  |29  |30  |31  |32  |33  |34  |35  |36  |37  |

|       | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |10  |11  |12  |13  |14  |15  |16  |17  |18  |19  |20  |21  |22  |23  |24  |25  |26  |27  |28  |29  |30  |31  |32  |33  |34  |35  |36  |37  |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $K(37, 9) = 8,$ |   |   |   |   |   |   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|       | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |10  |11  |12  |13  |14  |15  |16  |17  |18  |19  |20  |21  |22  |23  |24  |25  |26  |27  |28  |29  |30  |31  |32  |33  |34  |35  |36  |37  |
|       | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |10  |11  |12  |13  |14  |15  |16  |17  |18  |19  |20  |21  |22  |23  |24  |25  |26  |27  |28  |29  |30  |31  |32  |33  |34  |35  |36  |37  |

$K(36, 9) = 8,$

$K(37, 9) = 8,$
Appendix E

Particular Values of $K(n, k)$
Appendix E. Particular Values of $K(n,k)$

This appendix contains $(n,k)KSS$s which do not achieve the lower bound in Theorem 4.2 when $10 \leq k \leq 15$ for some $n$. Some of the minimal $(n,k)KSS$s constructed in this appendix appear in their array representation in Appendix G.

Lemma E.1.

(1) $K(20,10) = K(21,10) = K(22,10) = 5$.

(2) $K(24,10) = K(25,10) = K(26,10) = K(27,10) = 6$.

(3) $K(30,10) = K(31,10) = K(32,10) = K(33,10) = 7$.

(4) $K(36,10) = K(37,10) = K(38,10) = 8$.

Proof. (1) By Theorem 7.3, $K(20,10) \geq 4$, $K(21,10) \geq 4$ and $K(22,10) \geq 4$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ are a $(20,10)KSS$, a $(21,10)KSS$ and a $(22,10)KSS$ in 4 sets respectively. As $|\mathcal{K}_1|k = 40 < 46 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 40 < 49 \leq V(\mathcal{K}_2)$ and $|\mathcal{K}_3|k = 40 < 52 \leq V(\mathcal{K}_3)$ by Theorem 7.4, $K(20;10) > 4$, $K(21,10) > 4$ and $K(22,10) > 4$ by Corollary 7.3.

To form $(20,10)KSS$ in 5 sets use a minimal $(10,5)KSS$ on $[10]$ and a minimal $(10,5)KSS$ on $[20] \setminus [10]$ (see Theorem 7.7) with $M_1$.

To form a $(21,10)KSS$ in 5 sets use a minimal $(10,5)KSS$ on $[10]$ (see Theorem 7.7) and a minimal $(11,5)KSS$ on $[21] \setminus [10]$ (see Theorem 7.7) with $M_1$.

To form a $(22,10)KSS$ in 5 sets use a minimal $(10,4)KSS$ on $[10]$ (see Theorem 7.9) and a minimal $(12,6)KSS$ on $[22] \setminus [10]$ (see Lemma 8.1) with $M_1$.

Hence $K(20,10) = K(21,10) = K(22,10) = 5$.

(2) By Theorem 7.3, $K(24,10) \geq 5$, $K(25,1) \geq 5$, $K(26,10) \geq 5$ and $K(27,10) \geq 5$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ are a $(24,10)KSS$, a $(25,10)$, a $(26,10)KSS$ and a $(27,10)KSS$ in 5 sets respectively. As $|\mathcal{K}_1|k = 50 < 52 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 50 < 55 \leq V(\mathcal{K}_2)$, $|\mathcal{K}_3|k = 50 < 58 \leq V(\mathcal{K}_3)$ and $|\mathcal{K}_4|k = 50 < 61 \leq V(\mathcal{K}_4)$ by Theorem 7.4, $K(24,10) > 5$, $K(25,10) > 5$, $K(26,10) > 5$ and $K(27,10) > 5$ by
Appendix E. Particular Values of $K(n, k)$

Corollary 7.3.

To form a $(24, 10)KSS$ in 6 sets use a minimal $(10, 3)KSS$ on $[10]$ (see Theorem 7.10) and a minimal $(14, 7)KSS$ on $[24] \setminus [10]$ (see Lemma 8.10) with $M_1$.

To form a $(25, 10)KSS$ in 6 sets use a minimal $(10, 3)KSS$ on $[10]$ (see Theorem 7.10) and a minimal $(15, 7)KSS$ on $[25] \setminus [10]$ (see Lemma 8.10) with $M_1$.

To form a $(26, 10)KSS$ in 6 sets use a minimal $(10, 3)KSS$ on $[10]$ (see Theorem 7.10) and a minimal $(16, 7)KSS$ on $[26] \setminus [10]$ (see Lemma 8.2) with $M_1$.

To form a $(27, 10)KSS$ in 6 sets use a minimal $(10, 3)KSS$ on $[10]$ (see Theorem 7.10) and a minimal $(17, 7)KSS$ on $[27] \setminus [10]$ (see Lemma 8.2) with $M_1$.

Hence $K(24, 10) = K(25, 10) = K(26, 10) = K(27, 10) = 6$.

(3) By Theorem 7.3, $K(30, 10) \geq 6$, $K(31, 10) \geq 6 K(32, 10) \geq 6$ and $K(33, 10) \geq 6$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ are a $(30, 10)KSS$, a $(31, 10)$, a $(32, 10)KSS$ and a $(33, 10)KSS$ in 5 sets respectively. As $|\mathcal{K}_1|k = 60 < 63 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 60 < 66 \leq V(\mathcal{K}_2)$, $|\mathcal{K}_3|k = 60 < 69 \leq V(\mathcal{K}_3)$ and $|\mathcal{K}_4|k = 60 < 72 \leq V(\mathcal{K}_4)$ by Theorem 7.4, $K(30, 10) > 6$, $K(31, 10) > 6$, $K(32, 10) > 6$ and $K(33, 10) > 6$ by Corollary 7.3.

To form a $(30, 10)KSS$ in 7 sets use a minimal $(10, 3)KSS$ on $[10]$ (see Theorem 7.10) and a minimal $(20, 7)KSS$ on $[30] \setminus [10]$ (see Lemma 8.10) with $M_1$.

To form a $(31, 10)KSS$ in 7 sets use a minimal $(10, 3)KSS$ on $[10]$ (see Theorem 7.10) and a minimal $(21, 7)KSS$ on $[31] \setminus [10]$ (see Theorem 7.7) with $M_1$.

To form a $(32, 10)KSS$ in 7 sets use a minimal $(10, 3)KSS$ on $[10]$ (see Theorem 7.10) and a minimal $(22, 7)KSS$ on $[32] \setminus [10]$ (see Theorem 7.7) with $M_1$.

To form a $(33, 10)KSS$ in 7 sets use a minimal $(10, 3)KSS$ on $[10]$ (see Theorem 7.10) and a minimal $(23, 7)KSS$ on $[33] \setminus [10]$ (see Lemma 8.2) with $M_1$.

Hence $K(30, 10) = K(31, 10) = K(32, 10) = K(33, 10) = 7$. 

By Theorem 7.3, \( K(36, 10) \geq 7, \) \( K(37, 10) \geq 7 \) and \( K(38, 10) \geq 7. \) Assume \( K_1, K_2 \) and \( K_3 \) are a \((36, 10)KSS\), a \((37, 10)KSS\) and a \((38, 10)KSS\) in 7 sets respectively. As \(|K_1|k = 70 < 73 \leq V(K_1)|, \) \(|K_2|k = 70 < 76 \leq V(K_2)| \) and \(|K_3|k = 70 < 79 \leq V(K_3)| \) by Theorem 7.4, \( K(36, 10) > 7, \) \( K(37, 10) > 7 \) and \( K(38, 10) > 7 \) by Corollary 7.3.

To form a \((36, 10)KSS\) in 8 sets use a minimal \((10, 2)KSS\) on \([10]\) (see Theorem 7.8) and a minimal \((26, 8)KSS\) on \([36] \setminus [10]\) (see Lemma 8.11) with \( M_1 \).

To form a \((37, 10)KSS\) in 8 sets use a minimal \((10, 2)KSS\) on \([10]\) (see Theorem 7.8) and a minimal \((27, 8)KSS\) on \([37] \setminus [10]\) (see Lemma 8.11) with \( M_1 \).

To form a \((38, 10)KSS\) in 8 sets use a minimal \((10, 2)KSS\) on \([10]\) (see Theorem 7.8) and a minimal \((28, 8)KSS\) on \([38] \setminus [10]\) (see Theorem 7.7) with \( M_1 \).

Hence \( K(36, 10) = K(37, 10) = K(38, 10) = 8. \)

Note E.1. By Lemma 7.13, Theorems 7.5 to 7.7 and Lemmas 8.5 and E.1, the value \( K(n, 10) \) is known for all \( n \leq 40 \) except for \( 12 \leq n \leq 19. \) These values are determined in Chapter 9.

Lemma E.2.

1. \( K(22, 11) = K(23, 11) = K(24, 11) = 5. \)
4. \( K(38, 11) = K(39, 11) = K(40, 11) = 8. \)

Proof. (1) By Theorem 7.3, \( K(22, 11) \geq 4, \) \( K(23, 11) \geq 4 \) and \( K(24, 11) \geq 4. \) Assume \( K_1, K_2 \) and \( K_3 \) are a \((22, 11)KSS, \) a \((23, 11)KSS\) and a \((24, 11)KSS\) in 4 sets respectively. As \(|K_1|k = 44 < 52 \leq V(K_1)|, \) \(|K_2|k = 44 < 55 \leq V(K_2)| \) and \(|K_3|k = 44 < 58 \leq V(K_3)| \) by Theorem 7.4, \( K(22, 11) > 4, \) \( K(23, 11) > 4 \) and \( K(24, 11) > 4 \) by Corollary 7.3.
Appendix E. Particular Values of $K(n, k)$

To form a $(22, 11)KSS$ in 5 sets use a minimal $(11, 5)KSS$ on $[11]$ (see Theorem 7.7) and a minimal $(11, 6)KSS$ on $[22] \setminus [11]$ (see Lemma 8.1) with $M_1$.

To form a $(23, 11)KSS$ in 5 sets use a minimal $(11, 5)KSS$ on $[11]$ (see Theorem 7.7) and a minimal $(12, 6)KSS$ on $[23] \setminus [11]$ (see Lemma 8.1) with $M_1$.

To form a $(24, 11)KSS$ in 5 sets use a minimal $(10, 4)KSS$ on $[10]$ (see Theorem 7.9) and a minimal $(13, 7)KSS$ on $[24] \setminus [11]$ (see Lemma 8.2) with $M_2$.


(2) By Theorem 7.3, $K(26, 11) \geq 5$, $K(27, 11) \geq 5$, $K(28, 11) \geq 5$, $K(29, 11) \geq 5$ and $K(30, 11) \geq 5$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ and $\mathcal{K}_5$ are a $(26, 11)KSS$, a $(27, 11)KSS$, a $(28, 11)KSS$, a $(29, 11)KSS$ and a $(30, 11)KSS$ in 5 sets respectively. As $|\mathcal{K}_1|k = 55 < 58 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 55 < 61 \leq V(\mathcal{K}_2)$, $|\mathcal{K}_3|k = 55 < 64 \leq V(\mathcal{K}_3)$, $|\mathcal{K}_4|k = 55 < 67 \leq V(\mathcal{K}_4)$ and $|\mathcal{K}_5|k = 55 < 70 \leq V(\mathcal{K}_5)$ by Theorem 7.4, $K(26, 11) > 5$, $K(27, 11) > 5$, $K(28, 11) > 5$, $K(29, 11) > 5$, $K(29, 11) > 5$ and $K(30, 11) > 5$ by Corollary 7.3.

To form a $(26, 11)KSS$ in 6 sets use a minimal $(11, 4)KSS$ on $[11]$ (see Theorem 7.10) and a minimal $(15, 7)KSS$ on $[26] \setminus [11]$ (see Lemma 8.10) with $M_1$.

To form a $(27, 11)KSS$ in 6 sets use a minimal $(11, 4)KSS$ on $[11]$ (see Theorem 7.10) and a minimal $(16, 7)KSS$ on $[27] \setminus [11]$ (see Lemma 8.10) with $M_1$.

To form a $(28, 11)KSS$ in 6 sets use a minimal $(11, 4)KSS$ on $[11]$ (see Theorem 7.10) and a minimal $(17, 7)KSS$ on $[28] \setminus [11]$ (see Lemma 8.2) with $M_1$.

To form a $(29, 11)KSS$ in 6 sets use a minimal $(11, 4)KSS$ on $[11]$ (see Theorem 7.10) and a minimal $(18, 7)KSS$ on $[29] \setminus [11]$ (see Lemma 8.2) with $M_1$.

To form a $(30, 11)KSS$ in 6 sets use a minimal $(10, 3)KSS$ on $[10]$ (see Theorem 7.10) and a minimal $(19, 8)KSS$ on $[30] \setminus [11]$ (see Lemma 8.3) with $M_2$.

Appendix E. Particular Values of $K(n, k)$

(3) By Theorem 7.3, $K(32, 11) \geq 6$, $K(33, 11) \geq 6$, $K(34, 11) \geq 6$, $K(35, 11) \geq 6$ and $K(36, 11) \geq 6$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ and $\mathcal{K}_5$ are a $(32, 11)KSS$, a $(33, 11)KSS$, a $(34, 11)KSS$, a $(35, 11)KSS$ and a $(36, 11)KSS$ in 6 sets respectively. As $|\mathcal{K}_1|k = 66 < 69 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 66 < 72 \leq V(\mathcal{K}_2)$, $|\mathcal{K}_3|k = 66 < 75 \leq V(\mathcal{K}_3)$, $|\mathcal{K}_4|k = 66 < 78 \leq V(\mathcal{K}_4)$ and $|\mathcal{K}_5|k = 66 < 81 \leq V(\mathcal{K}_5)$ by Theorem 7.4, $K(32, 11) > 6$, $K(33, 11) > 6$, $K(34, 11) > 6$, $K(35, 11) > 6$ and $K(36, 11) > 6$ by Corollary 7.3.

To form a $(32, 11)KSS$ in 7 sets use a minimal $(11, 3)KSS$ on $[11]$ (see Theorem 7.10) and a minimal $(21, 8)KSS$ on $[32] \setminus [11]$ (see Lemma 8.11) with $M_1$.

To form a $(33, 11)KSS$ in 7 sets use a minimal $(11, 3)KSS$ on $[11]$ (see Theorem 7.10) and a minimal $(22, 8)KSS$ on $[33] \setminus [11]$ (see Lemma 8.11) with $M_1$.

To form a $(34, 11)KSS$ in 7 sets use a minimal $(11, 3)KSS$ on $[11]$ (see Theorem 7.10) and a minimal $(23, 8)KSS$ on $[34] \setminus [11]$ (see Lemma 8.3) with $M_1$.

To form a $(35, 11)KSS$ 7 sets use a minimal $(11, 3)KSS$ on $[11]$ (see Theorem 7.10) and a minimal $(24, 8)KSS$ on $[35] \setminus [11]$ (see Lemma 8.3) with $M_1$.

To form a $(36, 11)KSS$ 7 sets use a minimal $(11, 3)KSS$ on $[11]$ (see Theorem 7.10) and a minimal $(25, 8)KSS$ on $[36] \setminus [11]$ (see Lemma 8.3) with $M_1$.


(4) By Theorem 7.3, $K(38, 11) \geq 7$, $K(39, 11) \geq 7$, and $K(40, 11) \geq 7$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$ are a $(38, 11)KSS$, a $(39, 11)KSS$, and a $(40, 11)KSS$ in 7 sets respectively. As $|\mathcal{K}_1|k = 77 < 79 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 77 < 82 \leq V(\mathcal{K}_2)$, and $|\mathcal{K}_3|k = 77 < 85 \leq V(\mathcal{K}_3)$ by Theorem 7.4, $K(38, 11) > 7$, $K(39, 11) > 7$, and $K(40, 11) > 7$ by Corollary 7.3.

To form a $(38, 11)KSS$ in 8 sets use a minimal $(10, 2)KSS$ on $[10]$ (see Theorem 7.8) and a minimal $(27, 9)KSS$ on $[38] \setminus [11]$ (see Lemma 8.4) with $M_2$. 

Appendix E. Particular Values of $K(n, k)$

To form a $(39, 11)KSS$ in 8 sets use a minimal $(10, 2)KSS$ on $[10]$ (see Theorem 7.8) and a minimal $(28, 9)KSS$ on $[39] \setminus [11]$ (see Lemma 8.12) with $M_2$.

To form a $(40, 11)KSS$ in 8 sets use a minimal $(10, 2)KSS$ on $[10]$ (see Theorem 7.8) and a minimal $(29, 9)KSS$ on $[40] \setminus [11]$ (see Lemma 8.12) with $M_2$.

Hence $K(38, 11) = K(39, 11) = K(40, 11) = 8$. \hfill \Box

Note E.2. By Lemma 7.13, Theorems 7.5 to 7.7 and Lemmas 8.6 and E.2, the value $K(n, 11)$ is known for all $n \leq 40$ except for $13 \leq n \leq 21$. These values are determined in Chapter 9.

Lemma E.3.

\begin{enumerate}
\item $K(24, 12) = K(25, 12) = K(26, 12) = 5$.
\item $K(34, 12) = K(35, 12) = K(36, 12) = K(37, 12) = K(38, 12) = K(39, 12) = 7$.
\item $K(40, 12) = 8$.
\end{enumerate}

Proof. (1) By Theorem 7.3, $K(24, 12) \geq 4$, $K(25, 12) \geq 4$ and $K(26, 12) \geq 4$. Assume $K_1$, $K_2$ and $K_3$ are a $(24, 12)KSS$, a $(25, 12)KSS$ and a $(26, 12)KSS$ in 4 sets respectively. As $|K_1|k = 48 < 58 \leq V(K_1)$, $|K_2|k = 48 < 61 \leq V(K_2)$ and $|K_3|k = 48 < 64 \leq V(K_3)$ by Theorem 7.4, $K(24, 12) > 4$, $K(25, 12) > 4$ and $K(26, 12) > 4$ by Corollary 7.3.

To form a $(24, 12)KSS$ in 5 sets use two minimal $(12, 6)KSS$s on $[12]$ and $[24] \setminus [12]$ (see Lemma 8.1) with $M_1$.

To form a $(25, 12)KSS$ in 5 sets use a minimal $(11, 5)KSS$ on $[11]$ (see Theorem 7.7) and a minimal $(13, 7)KSS$ on $[25] \setminus [12]$ (see Lemma 8.2) with $M_2$. 

Appendix E. Particular Values of $K(n, k)$

To form a $(26, 12)KSS$ in 5 sets use a minimal $(11, 5)KSS$ on $[11]$ (see Theorem 7.7) and a minimal $(14, 7)KSS$ on $[26] \setminus [12]$ (see Lemma 8.2) with $M_2$.

Hence $K(24, 12) = K(25, 12) = K(26, 12) = 5$.

(2) By Theorem 7.3, $K(27, 12) \geq 5$, $K(28, 12) \geq 5$, $K(29, 12) \geq 5$, $K(30, 12) \geq 5$, $K(31, 12) \geq 5$ and $K(32, 12) \geq 5$. Assume $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5$ and $\mathcal{K}_6$ are a $(27, 12)KSS$, a $(28, 12)KSS$, a $(29, 12)KSS$, a $(30, 12)KSS$, a $(31, 12)KSS$ and a $(32, 12)KSS$ in 5 sets respectively. As $|\mathcal{K}_1|K = 60 < 61 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|K = 60 < 64 \leq V(\mathcal{K}_2)$, $|\mathcal{K}_3|K = 60 < 67 \leq V(\mathcal{K}_3)$, $|\mathcal{K}_4|K = 60 < 70 \leq V(\mathcal{K}_4)$, $|\mathcal{K}_5|K = 60 < 73 \leq V(\mathcal{K}_5)$ and $|\mathcal{K}_6|K = 60 < 76 \leq V(\mathcal{K}_6)$ by Theorem 7.4, by Corollary 7.3, $K(28, 12) > 5$, $K(29, 12) > 5$, $K(30, 12) > 5$, $K(31, 12) > 5$ and $K(32, 12) > 5$ by Corollary 7.3.

To form a $(27, 12)KSS$ in 6 sets use a minimal $(12, 5)KSS$ on $[12]$ (see Lemma 8.8) and a minimal $(15, 7)KSS$ on $[27] \setminus [12]$ (see Lemma 8.10) with $M_1$.

To form a $(28, 12)KSS$ in 6 sets use a minimal $(12, 4)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(16, 8)KSS$ on $[28] \setminus [12]$ (see Lemma 8.11) with $M_1$.

To form a $(29, 12)KSS$ in 6 sets use a minimal $(12, 4)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(17, 8)KSS$ on $[29] \setminus [12]$ (see Lemma 8.11) with $M_1$.

To form a $(30, 12)KSS$ in 6 sets use a minimal $(12, 4)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(18, 8)KSS$ on $[30] \setminus [12]$ (see Lemma 8.11) with $M_1$.

To form a $(31, 12)KSS$ in 6 sets use a minimal $(12, 4)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(19, 8)KSS$ on $[31] \setminus [12]$ (see Lemma 8.3) with $M_1$.

To form a $(32, 12)KSS$ in 6 sets use a minimal $(12, 4)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(20, 8)KSS$ on $[32] \setminus [12]$ (see Lemma 8.3) with $M_1$.

Appendix E. Particular Values of $K(n, k)$

(3) By Theorem 7.3, $K(34, 12) \geq 6$, $K(35, 12) \geq 6$, $K(36, 12) \geq 6$, $K(37, 12) \geq 6$, $K(38, 12) \geq 6$ and $K(39, 12) \geq 6$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_5$ and $\mathcal{K}_6$ are a $(34, 12)KSS$, a $(35, 12)KSS$, a $(36, 12)KSS$, a $(37, 12)KSS$, a $(38, 12)KSS$ and a $(39, 12)KSS$ in 6 sets respectively. As $|\mathcal{K}_1|k = 72 < 75 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 72 < 78 \leq V(\mathcal{K}_2)$, $|\mathcal{K}_3|k = 72 < 81 \leq V(\mathcal{K}_3)$, $|\mathcal{K}_4|k = 72 < 84 \leq V(\mathcal{K}_4)$, $|\mathcal{K}_5|k = 72 < 87 \leq V(\mathcal{K}_5)$ and $|\mathcal{K}_6|k = 72 < 90 \leq V(\mathcal{K}_6)$ by Theorem 7.4, $K(36, 12) > 6$, $K(37, 12) > 6$, $K(38, 12) > 6$ and $K(39, 12) > 6$ by Corollary 7.3.

To form a $(34, 12)KS$ in 7 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(22, 9)KSS$ on $[34] \setminus [12]$ (see Lemma 8.12) with $M_1$.

To form a $(35, 12)KS$ in 7 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(23, 9)KSS$ on $[35] \setminus [12]$ (see Lemma 8.12) with $M_1$.

To form a $(36, 12)KS$ in 7 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(24, 9)KSS$ on $[36] \setminus [12]$ (see Lemma 8.12) with $M_1$.

To form a $(37, 12)KS$ in 7 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(25, 9)KSS$ on $[37] \setminus [12]$ (see Lemma 8.12) with $M_1$.

To form a $(38, 12)KS$ in 7 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(26, 9)KSS$ on $[38] \setminus [12]$ (see Lemma 8.4) with $M_1$.

To form a $(39, 12)KS$ in 7 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(27, 9)KSS$ on $[39] \setminus [12]$ (see Lemma 8.4) with $M_1$.


(4) By Theorem 7.3, $K(40, 12) \geq 7$. Assume $\mathcal{K}_1$ is a $(40, 12)KSS$. As $|\mathcal{K}_1|k = 84 < 85 \leq V(\mathcal{K}_1)$ by Theorem 7.4, $K(40, 12) > 7$ by Corollary 7.3.

To form a $(40, 12)KSS$ in 8 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(28, 9)KSS$ on $[40] \setminus [12]$ (see Lemma 8.12) with $M_1$. 

Appendix E. Particular Values of $K(n, k)$

Hence $K(40, 12) = 8$. \qed

Note E.3. By Lemma 7.13, Theorems 7.5 to 7.7 and Lemmas 8.7 and E.3, the value $K(n, 12)$ is known for all $n \leq 40$ except for $14 \leq n \leq 23$. These values are determined in Chapter 9.

Lemma E.4.

1. $K(26, 13) = K(27, 13) = 5$

Proof. (1) By Theorem 7.3, $K(26, 13) \geq 4$ and $K(27, 13) \geq 4$. Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are a $(26, 13)KSS$ and a $(27, 13)KSS$ in 4 sets respectively. As $|\mathcal{K}_1|k = 52 < 64 \leq V(\mathcal{K}_1)$ and $|\mathcal{K}_2|k = 52 < 64 \leq V(\mathcal{K}_2)$ by Theorem 7.4, $K(26, 13) > 4$ and $K(27, 13) > 4$ by Corollary 7.3.

To form a $(26, 13)KSS$ in 5 sets use a minimal $(12, 6)KSS$ on $[12]$ (see Lemma 8.1) and a minimal $(13, 7)KSS$ on $[26] \setminus [13]$ (see Lemma 8.2) with $M_2$.

To form a $(27, 13)KSS$ in 5 sets use a minimal $(12, 6)KSS$ on $[12]$ (see Lemma 8.1) and a minimal $(14, 7)KSS$ on $[27] \setminus [13]$ (see Lemma 8.2) with $M_2$.

Hence $K(26, 13) = K(27, 13) = 5$.

(2) By Theorem 7.3, $K(29, 13) \geq 5$, $K(30, 13) \geq 5$, $K(31, 13) \geq 5$, $K(32, 13) \geq 5$, $K(33, 13) \geq 5$, $K(34, 13) \geq 5$ and $K(35, 13) \geq 5$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_5$, $\mathcal{K}_6$ and $\mathcal{K}_7$ are a $(29, 13)KSS$, a $(30, 13)KSS$, a $(31, 13)KSS$, a $(32, 13)KSS$, a $(33, 13)KSS$, a $(34, 13)KSS$ and a $(35, 13)KSS$ in 5 sets respectively. As $|\mathcal{K}_1|k = 65 < 67 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 65 < 70 \leq V(\mathcal{K}_2)$, $|\mathcal{K}_3|k = 65 < 73 \leq V(\mathcal{K}_3)$, $|\mathcal{K}_4|k = 65 < 76 \leq V(\mathcal{K}_4)$, $|\mathcal{K}_5|k = 65 < 79 \leq V(\mathcal{K}_5)$, $|\mathcal{K}_6|k = 65 < 82 \leq V(\mathcal{K}_6)$ and $|\mathcal{K}_7|k = 65 < 85 \leq V(\mathcal{K}_7)$ by Theorem 7.4, $K(29, 13) > 5$, $K(30, 13) > 5$, ...
Appendix E. Particular Values of $K(n, k)$

$K(31, 13) > 5$, $K(32, 13) > 5$, $K(33, 13) > 5$, $K(34, 13) > 5$ and $(K(35, 13) > 5$ by Corollary 7.3.

To form a $(29, 13)KSS$ in 6 sets use a minimal $(12, 5)KSS$ on $[12]$ (see Lemma 8.8) and a minimal $(16, 8)KSS$ on $[29] \setminus [13]$ (see Lemma 8.11) with $M_2$.

To form a $(30, 13)KSS$ in 6 sets use a minimal $(12, 5)KSS$ on $[12]$ (see Lemma 8.8) and a minimal $(17, 8)KSS$ on $[30] \setminus [13]$ (see Lemma 8.11) with $M_2$.

To form a $(31, 13)KSS$ in 6 sets use a minimal $(12, 5)KSS$ on $[12]$ (see Lemma 8.8) and a minimal $(18, 8)KSS$ on $[31] \setminus [13]$ (see Lemma 8.11) with $M_2$.

To form a $(32, 13)KSS$ in 6 sets use a minimal $(12, 5)KSS$ on $[12]$ (see Lemma 8.8) and a minimal $(19, 8)KSS$ on $[32] \setminus [13]$ (see Lemma 8.3) with $M_2$.

To form a $(33, 13)KSS$ in 6 sets use a minimal $(12, 5)KSS$ on $[12]$ (see Lemma 8.8) and a minimal $(20, 8)KSS$ on $[33] \setminus [13]$ (see Lemma 8.3) with $M_2$.

To form a $(34, 13)KSS$ in 6 sets use $M_3$ with 10 3-elements and 3 4-elements.

To form a $(35, 13)KSS$ in 6 sets use $M_3$ with 14 3-elements.


(3) By Theorem 7.3, $K(36, 13) \geq 6$, $K(37, 13) \geq 6$, $K(38, 13) \geq 6$, $K(39, 13) \geq 6$, and $K(40, 13) \geq 6$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$, and $\mathcal{K}_5$ are a $(36, 13)KSS$, a $(37, 13)KSS$, a $(38, 13)KSS$, a $(39, 13)KSS$, and a $(40, 13)KSS$ in 6 sets respectively. As $|\mathcal{K}_1|k = 78 < 81 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 78 < 84 \leq V(\mathcal{K}_2)$, $|\mathcal{K}_3|k = 78 < 87 \leq V(\mathcal{K}_3)$, $|\mathcal{K}_4|k = 78 < 90 \leq V(\mathcal{K}_4)$, and $|\mathcal{K}_5|k = 78 < 93 \leq V(\mathcal{K}_5)$ by Theorem 7.4, $K(36, 13) > 6$, $K(37, 13) > 6$, $K(38, 13) > 6$, $K(39, 13) > 6$, and $K(40, 13) > 6$ by Corollary 7.3.

To form a $(36, 13)KSS$ in 7 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(23, 10)KSS$ on $[36] \setminus [13]$ (see Lemma 8.5) with $M_2$. 


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To form a $(37, 13)KSS$ in 7 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(24, 10)KSS$ on $[37] \setminus [13]$ (see Lemma E.1) with $M_2$.

To form a $(38, 13)KSS$ in 7 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(25, 10)KSS$ on $[38] \setminus [13]$ (see Lemma E.1) with $M_2$.

To form a $(39, 13)KSS$ in 7 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(26, 10)KSS$ on $[39] \setminus [13]$ (see Lemma E.1) with $M_2$.

To form a $(40, 13)KSS$ in 7 sets use a minimal $(12, 3)KSS$ on $[12]$ (see Theorem 7.10) and a minimal $(27, 10)KSS$ on $[40] \setminus [13]$ (see Lemma E.1) with $M_2$.


Note E.4. By Lemma 7.13, Theorems 7.5 to 7.7 and Lemma E.4, the value $K(n, 13)$ is known for all $n \leq 40$ except for $15 \leq n \leq 25$. These values are determined in Chapter 9.

Lemma E.5.

(1) $K(28, 14) = 5$.


(3) $K(38, 14) = K(39, 14) = K(40, 14) = 7$.

Proof. (1) By Theorem 7.3, $K(28, 14) \geq 4$. Assume $\mathcal{K}$ is a $(28, 14)KSS$ in 4 sets. As $|\mathcal{K}|k = 56 < 70 \leq V(\mathcal{K})$ by Theorem 7.4, $K(28, 14) > 4$ Corollary 7.3.

To form a $(28, 14)KSS$ in 5 sets use a minimal $(13, 7)KSS$ on $[13]$ and a minimal $(14, 7)KSS$ on $[28] \setminus [14]$ (see Lemma 8.2) with $M_2$. Hence $K(28, 14) = 5$.

(2) By Theorem 7.3, $K(31, 14) \geq 5$, $K(32, 14) \geq 5$, $K(33, 14) \geq 5$, $K(34, 14) \geq 5$, $K(35, 14) \geq 5$, $K(36, 14) \geq 5$ and $K(37, 14) \geq 5$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_5$, $\mathcal{K}_6$ and $\mathcal{K}_7$ are a $(31, 14)KSS$, a $(32, 14)KSS$, a $(33, 14)KSS$, a $(34, 14)KSS$,
Appendix E. Particular Values of $K(n, k)$

a $(35,14)KSS$, a $(36,14)KSS$ and a $(37,14)KSS$ in 5 sets respectively. As $|\mathcal{K}_1|k = 70 < 73 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 70 < 76 \leq V(\mathcal{K}_2)$, $|\mathcal{K}_3|k = 70 < 79 \leq V(\mathcal{K}_3)$, $|\mathcal{K}_4|k = 70 < 82 \leq V(\mathcal{K}_4)$, $|\mathcal{K}_5|k = 70 < 85 \leq V(\mathcal{K}_5)$, $|\mathcal{K}_6|k = 70 < 88 \leq V(\mathcal{K}_6)$ and $|\mathcal{K}_7|k = 70 < 91 \leq V(\mathcal{K}_7)$, by Theorem 7.4, $K(31,14) > 5$, $K(32,14) > 5$, $K(33,14) > 5$, $K(34,14) > 5$, $K(35,14) > 5$, $K(36,14) > 5$ and $K(37,14) > 5$ by Corollary 7.3.

To form a $(31,14)KSS$ in 6 sets use a minimal $(13,6)KSS$ on $[13]$ (see Lemma 8.9) and a minimal $(17,8)KSS$ on $[31] \setminus [14]$ (see Lemma 8.11) with $M_2$.

To form a $(32,14)KSS$ in 6 sets use a minimal $(13,6)KSS$ on $[13]$ (see Lemma 8.9) and a minimal $(18,8)KSS$ on $[32] \setminus [14]$ (see Lemma 8.11) with $M_2$.

To form a $(33,14)KSS$ in 6 sets use a minimal $(13,6)KSS$ on $[13]$ (see Lemma 8.9) and a minimal $(19,8)KSS$ on $[33] \setminus [14]$ (see Lemma 8.11) with $M_2$.

To form a $(34,14)KSS$ in 6 sets use a minimal $(13,6)KSS$ on $[13]$ (see Lemma 8.9) and a minimal $(20,8)KSS$ on $[34] \setminus [14]$ (see Lemma 8.11) with $M_2$.

To form a $(35,14)KSS$ in 6 sets use a minimal $(13,5)KSS$ on $[13]$ (see Theorem 7.6) and a minimal $(21,9)KSS$ on $[35] \setminus [14]$ (see Lemma 8.4) with $M_2$.

To form a $(36,14)KSS$ in 6 sets use $M_3$ with 12 3-elements and 3 4-elements.

To form a $(37,14)KSS$ in 6 sets use $M_3$ with 16 3-elements.


(3) By Theorem 7.3, $K(38,14) \geq 6$, $K(39,14) \geq 6$, and $K(40,14) \geq 6$. Assume $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$ are a $(38,14)KSS$, a $(39,14)KSS$, and a $(40,14)KSS$ in 6 sets respectively. As $|\mathcal{K}_1|k = 84 < 87 \leq V(\mathcal{K}_1)$, $|\mathcal{K}_2|k = 84 < 90 \leq V(\mathcal{K}_2)$, and $|\mathcal{K}_3|k = 84 < 93 \leq V(\mathcal{K}_3)$ by Theorem 7.4, $K(38,14) > 6$, $K(39,14) > 6$, and $K(40,14) > 6$ by Corollary 7.3.
Appendix E. Particular Values of $K(n, k)$

To form a $(38, 14)KSS$ in 7 sets use $(13, 4)KSS$ on $[13]$ (see Theorem 7.10) and a minimal $(24, 10)KSS$ on $[38] \setminus [14]$ (see Lemma E.1) with $M_2$.

To form a $(39, 14)KSS$ in 7 sets use $(13, 4)KSS$ on $[13]$ (see Theorem 7.10) and a minimal $(25, 10)KSS$ on $[39] \setminus [14]$ (see Lemma E.1) with $M_2$.

To form a $(40, 14)KSS$ in 7 sets use $(13, 4)KSS$ on $[13]$ (see Theorem 7.10) and a minimal $(26, 10)KSS$ on $[40] \setminus [14]$ (see Lemma E.1) with $M_2$.

Hence $K(38, 14) = K(39, 14) = K(40, 14) = 7$.

Note E.5. By Lemma 7.13, Theorems 7.5 to 7.7 and Lemma E.5, the value $K(n, 14)$ is known for all $n \leq 40$ except for $16 \leq n \leq 27$. These values are determined in Chapter 9.

Lemma E.6.

(1) $K(30, 15) = 5$.


(3) $K(40, 15) = 7$.

Proof. (1) By Theorem 7.3, $K(30, 15) \geq 4$. Assume $\mathcal{K}$ is a $(30, 15)KSS$ in 4 sets. As $|\mathcal{K}|k = 60 < 76 \leq V(\mathcal{K})$ by Theorem 7.4, $K(30, 15) > 4$ by Corollary 7.3.

To form a $(30, 15)KSS$ in 5 sets use a minimal $(14, 7)KSS$ on $[14]$ (see Lemma 8.2) and a minimal $(15, 8)KSS$ on $[30] \setminus [15]$ (see Lemma 8.3) with $M_2$. Hence $K(30, 15) = 5$.

(2) By Theorem 7.3, $K(32, 15) \geq 4 K(33, 15) \geq 5 K(34, 15) \geq 5 K(35, 15) \geq 5 K(36, 15) \geq 5 K(37, 15) \geq 5 K(38, 15) \geq 5$ and $K(39, 15) \geq 5$. Assume $\mathcal{K}_1$ is a $(32, 15)KSS$ in 4 sets. As $|\mathcal{K}_1|k = 60 < 82 \leq V(\mathcal{K}_1)$ by Theorem 7.4, $K(32, 15) > 4$ by Corollary 7.3.
Appendix E. Particular Values of $K(n, k)$

Assume $K_1, K_2, K_3, K_4, K_5, K_7$ and $K_8$ are a $(32, 15)KSS$, a $(33, 15)KSS$, a $(34, 15)KSS$, a $(35, 15)KSS$, a $(36, 15)KSS$, a $(37, 15)KSS$, a $(38, 15)KSS$, a $(39, 15)KSS$ in 5 sets respectively.

As $|K_1|k = 75 < 76 \leq V(K_1)$, $|K_2|k = 75 < 79 \leq V(K_2)$, $|K_3|k = 75 < 82 \leq V(K_3)$, $|K_4|k = 75 < 85 \leq V(K_4)$, $|K_5|k = 75 < 88 \leq V(K_5)$, $|K_6|k = 75 < 91 \leq V(K_6)$, $|K_7|k = 75 < 94 \leq V(K_7)$ and $|K_8|k = 75 < 97 \leq V(K_8)$ by Theorem 7.4, $K(32, 15) > 5$, $K(33, 15) > 5$, $K(34, 15) > 5$, $K(35, 15) > 5$, $K(36, 15) > 5$, $K(37, 15) > 5$, $K(38, 15) > 5$ and $K(39, 15) > 5$ by Corollary 7.3.

To form a $(32, 15)KSS$ in 6 sets use a minimal $(14, 7)KSS$ on $[14]$ (see Lemma 8.2) and a minimal $(17, 8)KSS$ on $[32] \setminus [15]$ (see Lemma 8.11) with $M_2$. 

To form a $(33, 15)KSS$ in 6 sets use a minimal $(14, 7)KSS$ on $[14]$ (see Lemma 8.2) and a minimal $(18, 8)KSS$ on $[33] \setminus [15]$ (see Lemma 8.11) with $M_2$.

To form a $(34, 15)KSS$ in 6 sets use a minimal $(14, 7)KSS$ on $[14]$ (see Lemma 8.2) and a minimal $(19, 8)KSS$ on $[34] \setminus [15]$ (see Lemma 8.3) with $M_2$.

To form a $(35, 15)KSS$ in 6 sets use a minimal $(14, 7)KSS$ on $[14]$ (see Lemma 8.2) and a minimal $(20, 8)KSS$ on $[35] \setminus [15]$ (see Lemma 8.3) with $M_2$.

To form a $(36, 15)KSS$ in 6 sets use a minimal $(14, 5)KSS$ on $[14]$ (see Theorem 7.6) and a minimal $(21, 10)KSS$ on $[36] \setminus [15]$ (see Lemma E.1) with $M_2$.

To form a $(37, 15)KSS$ in 6 sets use a minimal $(14, 5)KSS$ on $[14]$ (see Theorem 7.6) and a minimal $(22, 10)K$ on $[37] \setminus [15]$ (see Lemma E.1) with $M_2$.

To form a $(38, 15)KSS$ in 6 sets use a minimal $(14, 5)KSS$ on $[14]$ (see Theorem 7.6) and a minimal $(23, 10)KSS$ on $[38] \setminus [15]$ (see Lemma 8.5) with $M_2$.

To form a $(39, 15)KSS$ in 6 sets use $M_3$ with 18 3-elements.

(3) By Theorem 7.3, $K(40, 15) \geq 5$. Assume $\mathcal{K}_1$ is a $(40, 15)KSS$ in 6 sets respectively. As $|\mathcal{K}_1|k = 90 < 93 \leq V(\mathcal{K}_1)$ by Theorem 7.4, $K(40, 15) > 6$ by Corollary 7.3.

To form a $(40, 15)KSS$ in 7 sets use a minimal $(15, 4)KSS$ on $[15]$ (see Theorem 7.10) and a minimal $(25, 11)KSS$ on $[40] \setminus [15]$ (see Lemma 8.6) with $M_1$.

Hence $K(40, 15) = 7$. \hfill \Box

Note E.6. By Lemma 7.13, Theorems 7.5 to 7.7 and Lemma E.6, the value $K(n, 15)$ is known for all $n \leq 40$ except for $17 \leq n \leq 29$. These values are determined in Chapter 9.
Appendix F

Examples of \((n, k)KSSs\) for
Section 8.3
### Appendix F. Examples of \((n, k)KSSs\) for Section 8.3

#### F.1  Case: \(k = 6\)

\[ K(11, 6) = 4, \]

1 2 3 4 5 6  
1 4 5 7 8 9  
2 5 6 7 8 10  
3 4 6 7 9 10 

#### F.2  Case: \(k = 7\)

\[ K(13, 7) = 4, \]

1 2 3 4 5 6 7  
1 2 3 4 8 11 12  
1 2 5 6 9 12 13  
1 3 5 7 10 11 13 

\[ K(14, 7) = 4, \]

1 2 3 4 5 6 7  
1 4 5 8 9 10 11  
2 5 6 8 9 12 13  
3 4 6 8 10 12 14
Appendix F. Examples of \((n,k)KSSs\) for Section 8.3

\[K(18,7) = 5,\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 5 & 8 & 9 & 10 & 11 \\
2 & 5 & 8 & 9 & 12 & 13 \\
3 & 6 & 8 & 10 & 12 & 14 \\
4 & 6 & 9 & 15 & 16 & 17 \\
\end{array}
\]

F.3 Case: \(k = 8\)

\[K(19,8) = 5,\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 5 & 6 & 9 & 10 & 11 \\
2 & 6 & 7 & 9 & 10 & 13 \\
3 & 7 & 8 & 9 & 11 & 13 \\
4 & 5 & 8 & 10 & 16 & 17 \\
\end{array}
\]

\[K(23,8) = 6,\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 6 & 9 & 10 & 11 & 12 \\
2 & 6 & 9 & 10 & 11 & 13 \\
3 & 6 & 10 & 12 & 13 & 14 \\
4 & 7 & 11 & 12 & 13 & 14 \\
5 & 7 & 9 & 19 & 20 & 21 \\
\end{array}
\]
Appendix F. Examples of \((n, k)KSSs\) for Section 8.3

F.4 Case: \(k = 9\)

\[ K(21, 9) = 6, \]

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 5 & 6 & 10 & 11 & 12 & 13 & 14 & 15 \\
2 & 6 & 7 & 10 & 13 & 14 & 16 & 19 & 20 \\
3 & 7 & 8 & 11 & 14 & 15 & 17 & 20 & 21 \\
4 & 5 & 8 & 12 & 13 & 15 & 18 & 19 & 21 \\
\end{array}
\]

F.5 Case: \(k = 10\)

\[ K(28, 10) = 6, \]

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 6 & 7 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
2 & 7 & 8 & 11 & 15 & 18 & 19 & 20 & 21 & 25 \\
3 & 8 & 9 & 12 & 15 & 18 & 19 & 22 & 23 & 26 \\
4 & 9 & 10 & 13 & 16 & 18 & 20 & 22 & 24 & 27 \\
5 & 6 & 10 & 14 & 16 & 19 & 25 & 26 & 27 & 28 \\
\end{array}
\]
Appendix F. Examples of \((n,k)KSSs\) for Section 8.3

\[ K(34,10) = 7, \]
\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 7 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
2 & 7 & 11 & 16 & 19 & 20 & 21 & 22 & 26 & 30 \\
3 & 8 & 12 & 16 & 19 & 20 & 23 & 24 & 27 & 31 \\
4 & 8 & 13 & 17 & 19 & 21 & 23 & 25 & 28 & 32 \\
5 & 9 & 14 & 17 & 20 & 26 & 27 & 28 & 29 & 33 \\
6 & 9 & 15 & 17 & 20 & 30 & 31 & 32 & 33 & 34 \\
\end{array}
\]

\[ K(39,10) = 8, \]
\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 8 & 11 & 12 & 13 & 14 & 18 & 22 & 27 & 33 \\
2 & 8 & 11 & 12 & 15 & 16 & 19 & 23 & 28 & 34 \\
3 & 8 & 11 & 13 & 15 & 17 & 20 & 24 & 29 & 35 \\
4 & 9 & 12 & 18 & 19 & 20 & 21 & 25 & 30 & 36 \\
5 & 9 & 12 & 22 & 23 & 24 & 25 & 26 & 31 & 37 \\
6 & 10 & 12 & 27 & 28 & 29 & 30 & 31 & 32 & 38 \\
7 & 10 & 12 & 33 & 34 & 35 & 36 & 37 & 38 & 39 \\
\end{array}
\]
F.6 Case: $k = 11$

$K(25, 11) = 5$, 

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
1 & 5 & 6 & 7 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
2 & 5 & 8 & 9 & 12 & 13 & 14 & 19 & 20 & 21 & 22 \\
3 & 6 & 8 & 10 & 12 & 15 & 16 & 19 & 20 & 23 & 24 \\
4 & 7 & 9 & 10 & 13 & 15 & 17 & 19 & 21 & 23 & 25 \\
\end{array}
\]
Appendix G

Examples of \((n, k)KSSs\) for Section 8.4
Appendix G. Examples of $(n, k)KSSs$ for Section 8.4

G.1 Case: $k = 6$

$K(13, 6) = 5,$

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 4 & 5 & 7 & 11 & 12 \\
2 & 4 & 6 & 8 & 11 & 13 \\
3 & 5 & 6 & 9 & 12 & 13 \\
3 & 5 & 6 & 10 & 12 & 13 \\
\end{array}
\]

$K(17, 6) = 6,$

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 7 & 8 & 9 & 10 & 14 \\
2 & 7 & 8 & 11 & 12 & 15 \\
3 & 7 & 9 & 11 & 13 & 16 \\
4 & 8 & 14 & 15 & 16 & 17 \\
5 & 8 & 14 & 15 & 16 & 17 \\
\end{array}
\]

G.2 Case: $k = 7$

$K(15, 7) = 5,$

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 4 & 5 & 8 & 9 & 10 & 13 \\
2 & 5 & 6 & 8 & 9 & 11 & 14 \\
3 & 4 & 6 & 8 & 10 & 12 & 15 \\
3 & 4 & 6 & 8 & 13 & 14 & 15 \\
\end{array}
\]
Appendix G. Examples of \((n, k)KSSs\) for Section 8.4

\[ K(19, 7) = 6, \]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 6 & 8 & 9 & 10 & 11 & 12 \\
2 & 6 & 8 & 13 & 14 & 15 & 16 \\
3 & 6 & 9 & 13 & 14 & 17 & 18 \\
4 & 7 & 10 & 13 & 15 & 17 & 19 \\
5 & 7 & 11 & 13 & 15 & 17 & 19 \\
\end{array}
\]

\[ K(24, 7) = 7, \]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 8 & 9 & 10 & 11 & 12 & 13 \\
2 & 8 & 14 & 15 & 16 & 17 & 21 \\
3 & 9 & 14 & 15 & 18 & 19 & 22 \\
4 & 10 & 14 & 16 & 18 & 20 & 23 \\
5 & 11 & 15 & 21 & 22 & 23 & 24 \\
6 & 12 & 15 & 21 & 22 & 23 & 24 \\
\end{array}
\]
Appendix G. Examples of \((n, k)\)KSSs for Section 8.4

G.3 Case: \(k = 8\)

\[ K(21, 8) = 6, \]

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 5 & 6 & 9 & 10 & 11 & 14 & 17 \\
2 & 6 & 7 & 9 & 10 & 12 & 15 & 18 \\
3 & 7 & 8 & 9 & 11 & 13 & 16 & 19 \\
4 & 5 & 8 & 9 & 14 & 15 & 16 & 20 \\
4 & 5 & 8 & 17 & 18 & 19 & 20 & 21 \\
\end{array}
\]

\[ K(26, 8) = 7, \]

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 7 & 9 & 10 & 11 & 14 & 17 & 22 \\
2 & 7 & 9 & 10 & 12 & 15 & 18 & 23 \\
3 & 7 & 9 & 11 & 13 & 16 & 19 & 24 \\
4 & 8 & 9 & 14 & 15 & 16 & 20 & 25 \\
5 & 8 & 17 & 18 & 19 & 20 & 21 & 26 \\
6 & 8 & 9 & 22 & 23 & 24 & 25 & 26 \\
\end{array}
\]
G.4 Case: \( k = 9 \)

\[ K(18, 9) = 5, \]

\[
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 6 & 10 & 11 & 12 & 13 & 14 \\
1 & 2 & 4 & 7 & 10 & 11 & 12 & 15 & 18 \\
1 & 3 & 5 & 8 & 10 & 11 & 13 & 16 & 18 \\
6 & 7 & 8 & 9 & 10 & 12 & 14 & 17 & 18
\]

\[ K(22, 9) = 6, \]

\[
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 6 & 7 & 10 & 11 & 12 & 13 & 14 & 15 \\
2 & 6 & 9 & 10 & 13 & 14 & 16 & 20 & 21 \\
3 & 7 & 8 & 11 & 13 & 15 & 17 & 20 & 22 \\
4 & 8 & 9 & 12 & 14 & 15 & 18 & 21 & 22 \\
5 & 8 & 9 & 12 & 14 & 15 & 19 & 21 & 22
\]
Appendix G. Examples of \((n, k)KSSs\) for Section 8.4

\[K(28, 9) = 7,\]
\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 7 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
2 & 7 & 10 & 14 & 17 & 18 & 19 & 20 & 21 \\
3 & 8 & 11 & 14 & 17 & 22 & 23 & 24 & 25 \\
4 & 8 & 12 & 15 & 18 & 22 & 23 & 26 & 27 \\
5 & 9 & 13 & 15 & 19 & 22 & 24 & 26 & 28 \\
6 & 9 & 13 & 15 & 20 & 22 & 24 & 26 & 28 \\
\end{array}
\]

\[K(34, 9) = 8,\]
\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 7 & 10 & 11 & 12 & 15 & 18 & 23 & 28 \\
2 & 7 & 10 & 11 & 13 & 16 & 19 & 24 & 29 \\
3 & 8 & 10 & 12 & 14 & 17 & 20 & 25 & 30 \\
4 & 8 & 10 & 15 & 16 & 17 & 21 & 26 & 31 \\
5 & 9 & 18 & 19 & 20 & 21 & 22 & 27 & 32 \\
6 & 9 & 10 & 23 & 24 & 25 & 26 & 27 & 33 \\
6 & 9 & 28 & 29 & 30 & 31 & 32 & 33 & 34 \\
\end{array}
\]
Appendix G. Examples of \((n, k)KSSs\) for Section 8.4

\[K(39, 9) = 9,\]

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
2 & 10 & 16 & 19 & 25 & 26 & 27 & 28 & 29 \\
3 & 11 & 16 & 20 & 25 & 30 & 31 & 32 & 33 \\
4 & 12 & 16 & 21 & 26 & 30 & 34 & 35 & 36 \\
5 & 13 & 18 & 22 & 27 & 31 & 34 & 37 & 38 \\
6 & 14 & 18 & 23 & 28 & 32 & 35 & 37 & 39 \\
7 & 15 & 18 & 24 & 29 & 33 & 36 & 38 & 39 \\
8 & 15 & 18 & 24 & 29 & 33 & 36 & 38 & 39 \\
\end{array}
\]

G.5 Case: \(k = 10\)

\[K(20, 10) = 5,\]

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 3 & 5 & 7 & 11 & 12 & 13 & 15 & 17 \\
1 & 2 & 4 & 5 & 8 & 11 & 12 & 14 & 15 & 18 \\
1 & 3 & 4 & 6 & 9 & 11 & 13 & 14 & 16 & 19 \\
2 & 3 & 4 & 6 & 10 & 12 & 13 & 14 & 16 & 20 \\
\end{array}
\]


Appendix G. Examples of \((n, k)KSSs\) for Section 8.4

\[K(24, 10) = 6,\]

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 6 & 7 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
2 & 6 & 10 & 11 & 12 & 13 & 18 & 19 & 20 & 21 \\
3 & 7 & 8 & 11 & 14 & 15 & 18 & 19 & 22 & 23 \\
4 & 8 & 9 & 12 & 14 & 16 & 18 & 20 & 22 & 24 \\
5 & 9 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 \\
\end{array}
\]

\[K(36, 10) = 8,\]

\[
\begin{array}{cccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 8 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
2 & 8 & 11 & 16 & 19 & 20 & 21 & 24 & 27 & 32 \\
3 & 9 & 12 & 16 & 19 & 20 & 22 & 25 & 28 & 33 \\
4 & 9 & 13 & 17 & 19 & 21 & 23 & 26 & 29 & 34 \\
5 & 10 & 14 & 17 & 19 & 24 & 25 & 26 & 30 & 35 \\
6 & 10 & 15 & 17 & 27 & 28 & 29 & 30 & 31 & 36 \\
7 & 10 & 15 & 17 & 19 & 32 & 33 & 34 & 35 & 36 \\
\end{array}
\]
Appendix G. Examples of \((n, k)KSSs\) for Section 8.4

G.6 Case: \(k = 11\)

\(K(22, 11) = 5,\)

\[
\begin{align*}
1 & \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \\
1 & \quad 2 \quad 3 \quad 4 \quad 8 \quad 12 \quad 13 \quad 14 \quad 15 \quad 18 \quad 22 \\
1 & \quad 2 \quad 5 \quad 6 \quad 9 \quad 12 \quad 13 \quad 14 \quad 16 \quad 19 \quad 22 \\
1 & \quad 3 \quad 5 \quad 7 \quad 10 \quad 13 \quad 15 \quad 16 \quad 17 \quad 20 \quad 22 \\
2 & \quad 8 \quad 9 \quad 10 \quad 11 \quad 14 \quad 15 \quad 16 \quad 17 \quad 21 \quad 22
\end{align*}
\]

\(K(26, 11) = 6,\)

\[
\begin{align*}
1 & \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \\
1 & \quad 6 \quad 7 \quad 8 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18 \\
2 & \quad 6 \quad 9 \quad 10 \quad 12 \quad 13 \quad 14 \quad 19 \quad 20 \quad 21 \quad 24 \\
3 & \quad 7 \quad 9 \quad 11 \quad 12 \quad 15 \quad 16 \quad 19 \quad 20 \quad 22 \quad 25 \\
4 & \quad 8 \quad 10 \quad 11 \quad 13 \quad 15 \quad 17 \quad 19 \quad 21 \quad 23 \quad 26 \\
5 & \quad 8 \quad 10 \quad 11 \quad 13 \quad 15 \quad 17 \quad 19 \quad 24 \quad 25 \quad 26
\end{align*}
\]

\(K(32, 11) = 7,\)

\[
\begin{align*}
1 & \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \\
1 & \quad 7 \quad 8 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18 \quad 19 \\
2 & \quad 7 \quad 11 \quad 12 \quad 16 \quad 17 \quad 20 \quad 21 \quad 22 \quad 25 \quad 28 \\
3 & \quad 8 \quad 9 \quad 13 \quad 16 \quad 18 \quad 20 \quad 21 \quad 23 \quad 26 \quad 29 \\
4 & \quad 9 \quad 10 \quad 14 \quad 17 \quad 18 \quad 20 \quad 22 \quad 24 \quad 27 \quad 30 \\
5 & \quad 10 \quad 11 \quad 15 \quad 17 \quad 18 \quad 20 \quad 25 \quad 26 \quad 27 \quad 31 \\
6 & \quad 10 \quad 11 \quad 15 \quad 17 \quad 18 \quad 28 \quad 29 \quad 30 \quad 31 \quad 32
\end{align*}
\]
Appendix G. Examples of $(n, k)KSS$ for Section 8.4

G.7 Case: $k = 12$

$K(24, 12) = 5,$

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<td>16</td>
<td>17</td>
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<td>22</td>
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</tbody>
</table>

$K(27, 12) = 6,$

<table>
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<th>5</th>
<th>6</th>
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<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
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<td>14</td>
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<td>14</td>
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</tr>
<tr>
<td>4</td>
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<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td>25</td>
<td>26</td>
<td>27</td>
</tr>
</tbody>
</table>
Appendix G. Examples of \((n, k)KSSs\) for Section 8.4

\[K(34, 12) = 7,\]

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
1 & 7 & 8 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\
2 & 8 & 9 & 13 & 18 & 19 & 22 & 23 & 24 & 25 & 26 & 27 \\
3 & 9 & 10 & 14 & 18 & 21 & 22 & 25 & 26 & 28 & 32 & 33 \\
5 & 11 & 12 & 16 & 20 & 21 & 24 & 26 & 27 & 30 & 33 & 34 \\
6 & 7 & 12 & 17 & 20 & 21 & 24 & 26 & 27 & 31 & 33 & 34 \\
\end{array}
\]

\[K(40, 12) = 8,\]

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
1 & 7 & 8 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\
2 & 8 & 9 & 13 & 19 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\
3 & 9 & 10 & 14 & 19 & 22 & 26 & 29 & 30 & 31 & 32 & 33 \\
4 & 10 & 11 & 15 & 20 & 23 & 26 & 29 & 34 & 35 & 36 & 37 \\
5 & 11 & 12 & 16 & 20 & 24 & 27 & 30 & 34 & 35 & 38 & 39 \\
6 & 7 & 12 & 17 & 21 & 25 & 27 & 31 & 34 & 36 & 38 & 40 \\
6 & 7 & 12 & 18 & 21 & 25 & 27 & 32 & 34 & 36 & 38 & 40 \\
\end{array}
\]
### Appendix G. Examples of \((n, k)\)KSSs for Section 8.4

#### G.8 Case: \(k = 13\)

**\(K(26, 13) = 5\),**

\[
\begin{array}{cccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
1 & 2 & 3 & 4 & 7 & 11 & 14 & 15 & 16 & 17 & 20 & 24 & 26 \\
1 & 2 & 3 & 5 & 8 & 12 & 14 & 15 & 16 & 18 & 21 & 25 & 26 \\
2 & 4 & 5 & 6 & 9 & 11 & 15 & 17 & 18 & 19 & 22 & 24 & 26 \\
3 & 4 & 5 & 6 & 10 & 12 & 16 & 17 & 18 & 19 & 23 & 25 & 26 \\
\end{array}
\]

**\(K(28, 13) = 6\),**

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
1 & 4 & 5 & 7 & 11 & 12 & 14 & 15 & 16 & 21 & 22 & 23 & 26 \\
2 & 4 & 6 & 8 & 11 & 13 & 14 & 17 & 18 & 21 & 22 & 24 & 27 \\
3 & 5 & 6 & 9 & 12 & 13 & 15 & 17 & 19 & 21 & 23 & 25 & 28 \\
3 & 5 & 6 & 10 & 12 & 13 & 15 & 17 & 19 & 21 & 26 & 27 & 28 \\
\end{array}
\]

**\(K(36, 13) = 7\),**

\[
\begin{array}{cccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
1 & 7 & 8 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 \\
2 & 8 & 9 & 14 & 15 & 16 & 17 & 24 & 25 & 26 & 31 & 32 & 33 \\
3 & 9 & 10 & 14 & 18 & 19 & 20 & 24 & 27 & 28 & 31 & 34 & 35 \\
\end{array}
\]
Appendix G. Examples of \((n, k)KSSs\) for Section 8.4

G.9 Case: \(k = 14\)

\(K(28, 14) = 5,\)

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
1 & 2 & 3 & 4 & 7 & 11 & 13 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\
1 & 2 & 3 & 5 & 8 & 12 & 13 & 15 & 16 & 17 & 22 & 23 & 24 & 25 \\
2 & 4 & 5 & 6 & 9 & 11 & 13 & 15 & 18 & 19 & 22 & 23 & 26 & 27 \\
3 & 4 & 5 & 6 & 10 & 12 & 13 & 16 & 18 & 20 & 22 & 24 & 26 & 28 \\
\end{array}
\]

\(K(29, 14) = 6,\)

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
1 & 2 & 3 & 4 & 5 & 6 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 29 \\
1 & 4 & 5 & 7 & 11 & 12 & 15 & 18 & 19 & 22 & 23 & 24 & 25 & 29 \\
2 & 4 & 6 & 8 & 11 & 13 & 16 & 18 & 20 & 22 & 23 & 26 & 27 & 29 \\
3 & 5 & 6 & 9 & 12 & 13 & 17 & 19 & 20 & 22 & 24 & 26 & 28 & 29 \\
3 & 5 & 6 & 10 & 12 & 13 & 17 & 19 & 20 & 22 & 24 & 26 & 28 & 29 \\
\end{array}
\]

\(K(31, 14) = 6,\)

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
1 & 2 & 3 & 4 & 5 & 6 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\
1 & 4 & 5 & 7 & 11 & 12 & 15 & 16 & 17 & 18 & 23 & 24 & 25 & 28 \\
3 & 5 & 6 & 9 & 12 & 13 & 15 & 17 & 19 & 21 & 23 & 25 & 27 & 30 \\
3 & 5 & 6 & 10 & 12 & 13 & 16 & 17 & 18 & 19 & 28 & 29 & 30 & 31 \\
\end{array}
\]
Appendix G. Examples of \((n, k)KSSs\) for Section 8.4

\[ K(36, 14) = 6, \]
\[
\begin{array}{cccccccccccccccc}
1 & 7 & 8 & 9 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 \\
3 & 8 & 11 & 12 & 14 & 18 & 19 & 20 & 24 & 27 & 28 & 31 & 34 & 35 \\
6 & 10 & 12 & 13 & 1 & 2 & 3 & 4 & 5 & 7 & 8 & 9 & 11 & 14 \\
\end{array}
\]

\[ K(38, 14) = 7, \]
\[
\begin{array}{ccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
1 & 2 & 3 & 4 & 5 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
3 & 8 & 11 & 12 & 16 & 20 & 24 & 25 & 26 & 27 & 32 & 33 & 34 & 35 \\
5 & 10 & 12 & 13 & 18 & 22 & 23 & 26 & 28 & 30 & 32 & 34 & 36 & 38 \\
6 & 10 & 12 & 13 & 19 & 23 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 \\
\end{array}
\]
Appendix G. Examples of \((n, k)KSSs\) for Section 8.4

G.10 Case: \(k = 15\)

\(K(36, 15) = 6,\)

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 2 & 3 & 6 & 10 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\
1 & 2 & 4 & 7 & 11 & 16 & 17 & 18 & 19 & 22 & 26 & 27 & 28 & 29 & 33 \\
1 & 3 & 5 & 8 & 12 & 16 & 17 & 18 & 20 & 23 & 26 & 27 & 30 & 31 & 34 \\
10 & 11 & 12 & 13 & 14 & 18 & 19 & 20 & 21 & 25 & 27 & 33 & 34 & 35 & 36 \\
\end{array}
\]

\(K(40, 15) = 7,\)

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 7 & 8 & 9 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \\
3 & 8 & 12 & 13 & 17 & 20 & 23 & 24 & 27 & 28 & 29 & 34 & 35 & 36 & 37 \\
4 & 9 & 12 & 14 & 18 & 21 & 23 & 25 & 27 & 30 & 31 & 34 & 35 & 38 & 39 \\
5 & 10 & 13 & 15 & 19 & 22 & 24 & 25 & 28 & 30 & 32 & 34 & 36 & 38 & 40 \\
6 & 11 & 14 & 15 & 19 & 22 & 24 & 25 & 28 & 30 & 32 & 34 & 36 & 38 & 40 \\
\end{array}
\]
Bibliography


